

Wave turbulence in incompressible Hall magnetohydrodynamics

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Abstract

We investigate the steepening of the magnetic fluctuation power law spectra observed in the inner solar wind for frequencies higher than 0.5 Hz. This high frequency part of the spectrum may be attributed to dispersive nonlinear processes. In that context, the long-time behavior of weakly interacting waves is examined in the framework of three-dimensional incompressible Hall magnetohydrodynamic (MHD) turbulence. The Hall term added to the standard MHD equations makes the Alfvén waves dispersive and circularly polarized. We introduce the generalized Elsässer variables and, using a complex helicity decomposition, we derive for three-wave interaction processes the general wave kinetic equations; they describe the nonlinear dynamics of Alfvén, whistler and ion cyclotron wave turbulence in the presence of a strong uniform magnetic field $B_0 \hat{\mathbf{e}}_{\parallel}$. Hall MHD turbulence is characterized by anisotropies of different strength: (i) for wavenumbers $kd_i \gg 1$ (d_i is the ion inertial length) nonlinear transfers are *essentially* in the direction perpendicular (\perp) to \mathbf{B}_0 ; (ii) for $kd_i \ll 1$ nonlinear transfers are *exclusively* in the perpendicular direction; (iii) for $kd_i \sim 1$, a moderate anisotropy is predicted. We show that electron and standard MHD turbulence can be seen as two frequency limits of the present theory but the standard MHD limit is singular; additionally, we analyze in detail the ion MHD turbulence limit. Exact power law solutions of the master wave kinetic equations are given in the small and large scale limits for which we have, respectively, the total energy spectra $E(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-5/2} |k_{\parallel}|^{-1/2}$ and $E(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-2}$. An anisotropic phenomenology is developed to describe continuously the different scaling laws of the energy spectrum;

one predicts $E(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-2} |k_{\parallel}|^{-1/2} (1 + k_{\perp}^2 d_i^2)^{-1/4}$. Nonlocal interactions between Alfvén, whistler and ion cyclotron waves are investigated; a non trivial dynamics exists only when a discrepancy from the equipartition between the large scale kinetic and magnetic energies happens.

1 Introduction

Spacecraft observations of the inner solar wind, *i.e.* for heliocentric distances less than 1AU, show magnetic and velocity fluctuations over a broad range of frequencies, from 10^{-5} Hz up to several hundred Hz (see *e.g.* Coleman, 1968; Belcher and Davis, 1971; Matthaeus and Goldstein, 1982; Roberts et al., 1987; Grappin et al., 1990; Burlaga, 1991; Leamon et al., 1998). These fluctuations possess many properties expected of fully developed weakly compressible magnetohydrodynamic (MHD) turbulence (Goldstein and Roberts, 1999). For that reason the interplanetary medium is often seen as a vast laboratory for studying many fundamental questions about turbulent plasmas.

Compressed interaction regions produced by velocity differences are clearly observed in the outer solar wind. In regions where fast streams overtake slow streams, the spectrum of the large scale magnetic field fluctuations follows a f^{-2} frequency power law (Burlaga et al., 1987) which was shown to be a spectral signature of jumps (Roberts et al., 1987). Compressive effects are however weaker in the inner solar wind. For example, the normalized density fluctuations near the current sheet at 0.3AU are often smaller than 5% (Bavassano et al., 1997). This tendency is confirmed by indirect measurements through radio wave interplanetary scintillation observations at heliocentric distances of 16 – 26 solar radius (Spangler, 2002). Waves and turbulence – the subject of this paper – are better observed in the pure/polar wind where generally the density fluctuations are weaker than in the current sheet. Therefore, the inner solar wind may be seen mainly as a weakly compressible medium.

The turbulent state of the solar wind was suggested by [24] who reported a power law behavior for energy spectra with spectral indices lying between -1 and -2 . The original signals being measured in time, these observational spectra are measured in frequency. Since the solar wind is supersonic and super-Alfvénic, the Taylor “frozen-in flow” hypothesis is usually used to connect directly a frequency to a wavenumber which allows even-

tually a comparison with theoretical predictions. Note, however, that for anisotropic turbulence a relevant comparison with theoretical predictions, like those made in this paper, are only possible if a three dimensional energy spectrum is accessible by *in situ* measurements, a situation that is not currently achieved (see, however, Matthaeus et al., 2005; see also the last Section). More precise measurements (see *e.g.* Matthaeus and Goldstein, 1982) revealed that the spectral index at low frequency is often about -1.7 which is closer to the Kolmogorov prediction (Kolmogorov, 1941) for neutral fluids ($-5/3$) rather than the Iroshnikov-Kraichnan prediction (Iroshnikov, 1963; Kraichnan, 1965) for magnetized fluids ($-3/2$). Both predictions are built, in particular, on the hypothesis of isotropic turbulence. However, the presence of Alfvén waves in the fast solar wind attests that the magnetic field has a preferential direction which is likely at the origin of anisotropic turbulence provided that the amount of counter-propagating Alfvén waves is enough. Indeed, *in situ* measurements of cross helicity show clearly that outward propagative Alfvén waves are the main component of the fast solar wind at short radial distances (Belcher and Davis, 1971) but they become less dominant beyond 1AU. Since pure Alfvén waves are exact solutions of the ideal incompressible MHD equations (see *e.g.* Pouquet, 1993), nonlinear interactions should be suppressed if only one type of waves is present. The variance analysis of the magnetic field components and magnitude shows clearly that the magnetic field vector of the fast solar wind has a varying direction but with only a weak variation in magnitude (see *e.g.* Forsyth et al., 1996ab). Typical values give a ratio of the normalized variance of the field magnitude smaller than 10% whereas for the components it can be as large as 50%. In these respects, the interplanetary magnetic field may be seen as a vector lying approximately around the Parker spiral direction with only weak magnitude variations (Barnes, 1981). Solar wind anisotropy with more power perpendicular to the mean magnetic field than that parallel is pointed out by data analyses (Belcher and Davis, 1971; Klein et al., 1993) with a ratio of power up to 30. From single-point spacecraft measurements it is however not possible to specify the exact three-dimensional form of the spectral tensor of the magnetic or velocity fluctuations. In absence of such data, [15] with a quasi two-dimensional model, in which wave vectors are nearly perpendicular to the large-scale magnetic field, argued that about 85% of solar wind turbulence possesses a dominant 2D component. Additionally, solar wind anisotropies is detected through radio wave scintillations which reveal that density spectra close to the Sun are highly anisotropic with irregularities

stretched out mainly along the radial direction (Armstrong et al., 1990).

Most of the papers dealing with interplanetary turbulence tend to focus on a frequency inertial range where the MHD approximation is well satisfied. It is the domain where the power spectral index is often found to be around the Kolmogorov index. During the last decades several properties of the solar wind turbulence have been understood in this framework. Less understood is what happens outside of this range of frequencies. At lower frequencies ($< 10^{-5}\text{Hz}$) flatter power laws are found in particular for the fast solar wind with indices close to -1 and even less (in absolute value) for the smallest frequencies. These large scales are often interpreted as the energy containing scales. But the precise role of the low solar corona in the generation of such scales and the origin of the evolution of the spectral index when the distance from the Sun increases are still not well understood (see *e.g.* Velli et al., 1989; Horbury, 1999). For frequencies higher than 0.5Hz a steepening of the magnetic fluctuation power law spectra is observed over more than two decades (Coroniti et al., 1982; Denskat et al., 1983, Leamon et al., 1998) with a spectral index on average around -3 . Note that the latest analysis made with the Cluster spacecraft data reveals a less steep spectral index about -2.12 (Bale et al., 2005). This new range, exhibiting a power law, is characterized by a bias of the polarization suggesting that these fluctuations are likely to be right-hand polarized, outward propagating waves (Goldstein et al., 1994). Various indirect lines of evidence indicate that these waves propagate at large angles to the background magnetic field and that the power in fluctuations parallel to the background magnetic field is much less than the perpendicular one (Coroniti et al., 1982; Leamon et al., 1998). For these reasons, it is thought (*e.g.* Stawicki et al., 2001) that Alfvén – left circularly polarized – fluctuations are suppressed by proton cyclotron damping and that the high frequency power law spectra are likely to consist of whistler waves. This scenario proposed is supported by direct numerical simulations of compressible $2\frac{1}{2}\text{D}$ Hall MHD turbulence (Ghosh et al., 1996) where a steepening of the spectra is found and associated with the appearance of right circularly polarized fluctuations. It is plausible that what has been conventionally thought of as a dissipation range is actually a dispersive or inertial range and that the steeper power law may be due to nonlinear wave processes rather than dissipation (see *e.g.* Krishan and Mahajan, 2004). Under this new interpretation, the resistive dissipation range of frequencies may be moved to frequencies higher than the electron cyclotron frequency. A recent study (Stawicki et al., 2001) suggests that the treatment of the solar

wind dispersive range should include magnetosonic/whistler waves since it is often observed that the high frequency fluctuations of the magnetic field are much smaller than the background magnetic field (see also Forsyth et al., 1996ab). It seems therefore that a nonlinear theory built on weak wave turbulence may be a useful point of departure for understanding the detailed physics of solar wind turbulence.

It is well known that the presence of a mean magnetic field plays a fundamental role in the behavior of compressible or incompressible MHD turbulence (Montgomery and Turner, 1981; Shebalin et al. 1983; Oughton et al., 1994; Matthaeus et al., 1996; Kinney and McWilliams, 1998; Cho and Vishniac, 2000; Oughton and Matthaeus, 2005). The main effect is that the mean magnetic field renders the turbulence quasi-bidimensional with a nonlinear transfer essentially perpendicular to its direction. Note that in such a situation, it is known that compressible MHD behaves very similarly to reduced MHD (see *e.g.* Dmitruk et al, 2005). This property is generalized to $2\frac{1}{2}$ D compressible Hall MHD for high and low beta β plasma simulations (Ghosh and Goldstein, 1997) for which strong anisotropies are also found when a strong mean magnetic field is present. It was suggested that the action of the Hall term is to provide additional suppression of energy cascades along the mean field direction and incompressibility may not be able to reproduce the dynamics seen in the simulations. This conclusion is slightly different to what we find in the present study where we show that incompressibility in Hall MHD allows to reproduce several properties observed like anisotropic turbulence. Incompressible Hall MHD is often used to understand, for example, the main impact of the Hall term on flowing plasmas (Ohsaki, 2005), or in turbulent dynamo (Mininni et al, 2003). In the present paper, we derive a weak wave turbulence formalism for incompressible Hall MHD in the presence of a strong external mean magnetic field, where Alfvén, ion cyclotron and whistler/electron waves are taken into account. One of the main results is the derivation of the wave kinetic equations at the lowest order, *i.e.* for three-wave interaction processes. For such a turbulence, it is possible to show, in particular, a global tendency towards anisotropy with nonlinear transfers preferentially in the direction perpendicular to the external magnetic field. Hall MHD wave turbulence theory describes a wide range of frequencies, from the low frequency limit of pure Alfvénic turbulence to the high frequency limit of whistler wave turbulence for which the asymptotic theories have been derived recently (Galtier et al., 2000-2002; Galtier and Bhattacharjee, 2003). By recovering both theories as two particular lim-

its, we recover all the well-known properties associated. Ion cyclotron wave turbulence appears as a third particular limit for which we report a detailed analysis. The energy spectrum of Hall MHD is characterized by two inertial ranges, which are exact solutions of the wave kinetic equations, separated by a knee. The position of the knee corresponds to the scale where the Hall term becomes sub/dominant. We develop a single anisotropic phenomenology that recovers the power law solutions found and makes the link continuously in wavenumbers between the two scaling laws. A nonlocal analysis performed on the wave kinetic equations reveals that a non trivial dynamics between Alfvén, whistler and ion cyclotron waves happens only when a discrepancy from the equipartition between the large scale kinetic and magnetic energies exists. We believe that the description given here may help to better understand the inner solar wind observations and, in particular, the existence of dispersive/non-dispersive inertial ranges.

The organization of the paper is as follows: in Section 2, we discuss about the approximation of incompressible Hall MHD and the existence of transverse, circularly polarized waves; we introduce the generalized Elsässer variables and the complex helical decomposition. In Section 3, we develop the wave turbulence formalism and we derive the wave kinetic equations for three-wave interaction processes. Section 4 is devoted to the general properties of Hall MHD wave turbulence. Section 5 deals with the small and large scale limits respectively of the master equations of wave turbulence. In particular, we give a detail study of the ion cyclotron wave turbulence. In Section 6, we derive an anisotropic heuristic description which, in particular, makes the link between the previous predictions. In Section 7, we analyze nonlocal interactions between Alfvén, whistler and ion cyclotron waves. Section 8 is devoted to the possible sources of anisotropy. In Section 9, we discuss about the domain of validity of the theory and the difference between wave and strong turbulence. Finally, we conclude with a summary and a general discussion in the last Section.

2 Hall magnetohydrodynamics approximation

2.1 Generalized Ohm's law

Hall MHD is an extension of the standard MHD where the ion inertia is retained in Ohm's law. The generalized Ohm's law, in SI unit, is then given

by

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} - \frac{\mathbf{J} \times \mathbf{B}}{n e} = \mu_0 \eta \mathbf{J}, \quad (1)$$

where \mathbf{E} is the electric field, \mathbf{V} is the plasma flow velocity, \mathbf{B} is the magnetic field, \mathbf{J} is the current density, n is the electron density, e is the magnitude of the electron charge, μ_0 is the permeability of free space and η is the magnetic diffusivity. The Hall effect, represented by the last term in the left hand side of the generalized Ohm's law, becomes relevant when we intend to describe the plasma dynamics up to length scales shorter than the ion inertial length d_i ($d_i = c/\omega_{pi}$, where c is the speed of light and ω_{pi} is the ion plasma frequency) and time scales of the order, or shorter, than the ion cyclotron period ω_{ci}^{-1} . It is one of the most important manifestations of the velocity difference between electrons and ions when kinetic effects are not taken into account. The importance of the Hall effect in astrophysics has been pointed out to understand, for example, the presence of instabilities in protostellar disks (Balbus and Terquem, 2001), the magnetic field evolution in neutron star crusts (Goldreich and Reisenegger, 1992; Cumming et al, 2004), impulsive magnetic reconnection (see *e.g.* Bhattacharjee, 2004) or the formation of filaments (see *e.g.* Passot and Sulem, 2003; Dreher et al., 2005).

2.2 Incompressible Hall MHD equations

The inclusion of the Hall effect in the Ohm's law leads, in the incompressible case, to the following Hall MHD equations:

$$\nabla \cdot \mathbf{V} = 0, \quad (2)$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P_* + \mathbf{B} \cdot \nabla \mathbf{B} + \nu \nabla^2 \mathbf{V}, \quad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V} - d_i \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}, \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (5)$$

where \mathbf{B} has been normalized to a velocity ($\mathbf{B} \rightarrow \sqrt{\mu_0 n m_i} \mathbf{B}$, with m_i the ion mass), P_* is the total (magnetic plus kinetic) pressure and ν is the viscosity. The Hall effect appears in the induction equation as an additional term proportional to the ion inertial length d_i which means that it is effective when the dynamical scale is small enough. In other words, for large scale phenomena this term is negligible and we recover the standard MHD equations. In the

opposite limit, *e.g.* for very fast time scales ($\ll \omega_{ci}^{-1}$), ions do not have time to follow electrons and they provide a static homogeneous background on which electrons move. Such a model where the dynamics is entirely governed by electrons is called the Electron MHD (EMHD) model (Kingsep et al., 1990; Shukla and Stenflo, 1999). It can be recovered from Hall MHD by taking the limits of small velocity \mathbf{V} and large d_i . The Electron and Hall MHD approximations are particularly relevant in the context of collisionless magnetic reconnection where the diffusion region develops multiscale structures corresponding to ion and electron characteristic lengths (Huba, 1995; Biskamp, 1997). For example, it is often considered that whistler/EMHD turbulence may act as a detector for magnetic reconnection at the magnetopause (Cai et al., 2001).

2.3 Three-dimensional inviscid invariants

The three inviscid ($\nu = \eta = 0$) quadratic invariants of incompressible Hall MHD are the total energy

$$E = \frac{1}{2} \int (\mathbf{V}^2 + \mathbf{B}^2) d\mathcal{V}, \quad (6)$$

the magnetic helicity

$$H_m = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{B} d\mathcal{V}, \quad (7)$$

and the generalized hybrid helicity

$$H_G = \frac{1}{2} \int (\mathbf{A} + d_i \mathbf{V}) \cdot (\mathbf{B} + d_i \nabla \times \mathbf{V}) d\mathcal{V}, \quad (8)$$

with \mathbf{A} the vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$). The third invariant generalizes the cross-helicity, $H_c = (1/2) \int \mathbf{V} \cdot \mathbf{B} d\mathcal{V}$, which is not conserved anymore when the Hall term is present in the MHD equations. The generalized hybrid helicity can be seen as the product of two generalized quantities, a vector potential $\mathbf{\Upsilon} = \mathbf{A} + d_i \mathbf{V}$ and a vorticity $\mathbf{\Omega} = \mathbf{B} + d_i \nabla \times \mathbf{V}$. The role played by the generalized vorticity is somewhat equivalent to the one played by the magnetic field in standard MHD (Woltjer, 1958). Indeed, both quantities obey the same Lagrangian equation, which is for $\mathbf{\Omega}$,

$$\frac{d\mathbf{\Omega}}{dt} = \mathbf{\Omega} \cdot \nabla \mathbf{V}. \quad (9)$$

By applying the Helmholtz's law (see *e.g.* Davidson, 2001) to Hall MHD, we see that the generalized vorticity lines are frozen into the plasma (see *e.g.* Sahraoui et al., 2003). The presence of the Hall term breaks such a property for the magnetic field which is however still frozen but only in the electron flow: the introduction of the electron velocity \mathbf{V}_e , with $\mathbf{V} \times \mathbf{B} - \mathbf{J} \times \mathbf{B}/ne \simeq \mathbf{V}_e \times \mathbf{B}$, leads to

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{V}_e, \quad (10)$$

which proves the statement. We will see below that the detailed conservation of invariants is the first test that the wave kinetic equations have to satisfy.

2.4 Incompressible Hall MHD waves

One of the main effects produced by the presence of the Hall term is that the linearly polarized Alfvén waves, solutions of the standard MHD equations, become circularly polarized and dispersive (see *e.g.* Mahajan and Krishan, 2005; Sahraoui et al., 2005). Indeed, if we linearize equations (2)–(5) around a strong uniform magnetic field \mathbf{B}_0 such that,

$$\mathbf{B}(\mathbf{x}) = B_0 \hat{\mathbf{e}}_{\parallel} + \epsilon \mathbf{b}(\mathbf{x}), \quad (11)$$

$$\mathbf{V}(\mathbf{x}) = \epsilon \mathbf{v}(\mathbf{x}), \quad (12)$$

with ϵ a small parameter ($0 < \epsilon \ll 1$), \mathbf{x} a three-dimensional displacement vector, and $\hat{\mathbf{e}}_{\parallel}$ a unit vector ($|\hat{\mathbf{e}}_{\parallel}| = 1$), then we obtain the following inviscid equations in Fourier space:

$$\mathbf{k} \cdot \mathbf{v}_{\mathbf{k}} = 0, \quad (13)$$

$$\partial_t \mathbf{v}_{\mathbf{k}} - ik_{\parallel} B_0 \mathbf{b}_{\mathbf{k}} = \epsilon \{ -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla P_* + \mathbf{b} \cdot \nabla \mathbf{b} \}_{\mathbf{k}}, \quad (14)$$

$$\partial_t \mathbf{b}_{\mathbf{k}} - ik_{\parallel} B_0 \mathbf{u}_{\mathbf{k}} - d_i B_0 k_{\parallel} \mathbf{k} \times \mathbf{b}_{\mathbf{k}} = \epsilon \{ -\mathbf{v} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{v} - d_i \nabla \times [(\nabla \times \mathbf{b}) \times \mathbf{b}] \}_{\mathbf{k}}, \quad (15)$$

$$\mathbf{k} \cdot \mathbf{b}_{\mathbf{k}} = 0, \quad (16)$$

where the wavevector $\mathbf{k} = k \hat{\mathbf{e}}_k = \mathbf{k}_{\perp} + k_{\parallel} \hat{\mathbf{e}}_{\parallel}$ ($k = |\mathbf{k}|$, $k_{\perp} = |\mathbf{k}_{\perp}|$, $|\hat{\mathbf{e}}_k| = 1$) and $i^2 = -1$. The index \mathbf{k} denotes the Fourier transform, defined by the relation

$$\mathbf{v}(\mathbf{x}) \equiv \int \mathbf{v}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad (17)$$

where $\mathbf{v}(\mathbf{k}) = \mathbf{v}_{\mathbf{k}} = \tilde{\mathbf{v}}_{\mathbf{k}} e^{-i\omega t}$ (the same notation is used for the magnetic field). The linear dispersion relation ($\epsilon = 0$) reads

$$\omega^2 - (\Lambda d_i B_0 k_{\parallel} k) \omega - B_0^2 k_{\parallel}^2 = 0, \quad (18)$$

with

$$\begin{Bmatrix} \tilde{\mathbf{v}}_{\mathbf{k}} \\ \tilde{\mathbf{b}}_{\mathbf{k}} \end{Bmatrix} = \Lambda i \hat{\mathbf{e}}_k \times \begin{Bmatrix} \tilde{\mathbf{v}}_{\mathbf{k}} \\ \tilde{\mathbf{b}}_{\mathbf{k}} \end{Bmatrix}. \quad (19)$$

We obtain the solutions

$$\omega \equiv \omega_{\Lambda}^s = \frac{sk_{\parallel}kd_iB_0}{2} \left(s\Lambda + \sqrt{1 + \frac{4}{d_i^2k^2}} \right), \quad (20)$$

where the value (± 1) of s defines the directional wave polarity. In other words, we have $sk_{\parallel} \geq 0$ and ω_{Λ}^s is a positive frequency. The Alfvén wave polarization Λ tells us if the wave is right ($\Lambda = s$) or left ($\Lambda = -s$) circularly polarized. In the first case, we are dealing with whistler waves, whereas in the latter case with ion cyclotron waves. We see that the transverse circularly polarized Alfvén waves are dispersive and we note that we recover the two well-known limits, *i.e.* the pure whistler waves ($\omega = sk_{\parallel}kd_iB_0$) in the high frequency limit ($kd_i \rightarrow \infty$), and the standard Alfvén waves ($\omega = sk_{\parallel}B_0$) in the low frequency limit ($kd_i \rightarrow 0$). The Alfvén waves become linearly polarized only when the Hall term vanishes: when the Hall term is present, whatever its magnitude is, the Alfvén waves are circularly polarized. Note that this situation is different from the compressible case for which the Alfvén waves are elliptically polarized. As expected, it is possible to show (Hameiri et al., 2005; Sahraoui et al., 2005) that the ion cyclotron wave has a resonance at the frequency $\omega_{ci}k_{\parallel}/k$, where $\omega_{ci} = B_0/d_i$. Therefore, with such an approximation, only whistler waves survive at high frequency.

2.5 Complex helicity decomposition

Given the incompressibility constraints (13) and (16), it is convenient to project the Hall MHD equations in the plane orthogonal to \mathbf{k} . We will use the complex helicity decomposition technique which has been shown to be effective in providing a compact description of the dynamics of 3D incompressible fluids (Craya, 1958; Moffatt, 1970; Kraichnan, 1973; Cambon et al., 1989; Lesieur, 1990; Waleffe, 1992; Turner, 2000; Galtier, 2003; Galtier and Bhattacharjee, 2003). The complex helicity basis is also particularly useful

since it allows to diagonalize systems dealing with circularly polarized waves. We introduce the complex helicity decomposition

$$\mathbf{h}^\Lambda(\mathbf{k}) \equiv \mathbf{h}_\mathbf{k}^\Lambda = \hat{\mathbf{e}}_\theta + i\Lambda\hat{\mathbf{e}}_\Phi, \quad (21)$$

where

$$\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\Phi \times \hat{\mathbf{e}}_k, \quad (22)$$

$$\hat{\mathbf{e}}_\Phi = \frac{\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_k}{|\hat{\mathbf{e}}_\parallel \times \hat{\mathbf{e}}_k|}, \quad (23)$$

and $|\hat{\mathbf{e}}_\theta(\mathbf{k})|=|\hat{\mathbf{e}}_\Phi(\mathbf{k})|=1$. We note that $(\hat{\mathbf{e}}_k, h_\mathbf{k}^+, h_\mathbf{k}^-)$ form a complex basis with the following properties:

$$\mathbf{h}_\mathbf{k}^{-\Lambda} = \mathbf{h}_{-\mathbf{k}}^\Lambda, \quad (24)$$

$$\hat{\mathbf{e}}_k \times \mathbf{h}_\mathbf{k}^\Lambda = -i\Lambda \mathbf{h}_\mathbf{k}^\Lambda, \quad (25)$$

$$\mathbf{k} \cdot \mathbf{h}_\mathbf{k}^\Lambda = 0, \quad (26)$$

$$\mathbf{h}_\mathbf{k}^\Lambda \cdot \mathbf{h}_\mathbf{k}^{\Lambda'} = 2\delta_{-\Lambda'\Lambda}. \quad (27)$$

We project the Fourier transform of the original vectors $\mathbf{v}(\mathbf{x})$ and $\mathbf{b}(\mathbf{x})$ on the helicity basis:

$$\mathbf{v}_\mathbf{k} = \sum_\Lambda \mathcal{U}_\Lambda(\mathbf{k}) \mathbf{h}_\mathbf{k}^\Lambda = \sum_\Lambda \mathcal{U}_\Lambda \mathbf{h}_\mathbf{k}^\Lambda, \quad (28)$$

$$\mathbf{b}_\mathbf{k} = \sum_\Lambda \mathcal{B}_\Lambda(\mathbf{k}) \mathbf{h}_\mathbf{k}^\Lambda = \sum_\Lambda \mathcal{B}_\Lambda \mathbf{h}_\mathbf{k}^\Lambda. \quad (29)$$

We introduce expressions of the fields into the Hall MHD equations written in Fourier space and we multiply by vector $\mathbf{h}_{-\mathbf{k}}^\Lambda$. First, we will focus on the linear dispersion relation ($\epsilon = 0$) which reads:

$$\partial_t \mathcal{Z}_\Lambda^s = -i\omega_\Lambda^s \mathcal{Z}_\Lambda^s, \quad (30)$$

with

$$\mathcal{Z}_\Lambda^s \equiv \mathcal{U}_\Lambda + \xi_\Lambda^s \mathcal{B}_\Lambda, \quad (31)$$

$$\xi_\Lambda^s(k) = \xi_\Lambda^s = -\frac{sd_i k}{2} \left(s\Lambda + \sqrt{1 + \frac{4}{d_i^2 k^2}} \right). \quad (32)$$

Equation (30) shows that \mathcal{Z}_Λ^s are the “good” variables for our system. These eigenvectors combine the velocity and the magnetic field in a non trivial way by a factor ξ_Λ^s (with $\omega_\Lambda^s = -B_0 k_\parallel \xi_\Lambda^s$). In the large scale limit ($kd_i \rightarrow 0$),

we see that $\xi_\Lambda^s \rightarrow -s$; we recover the Elsässer variables used in standard MHD. In the small scale limit ($kd_i \rightarrow \infty$), we have $\xi_\Lambda^s \rightarrow -s d_i k$, for $\Lambda = s$ (whistler waves), or $\xi_\Lambda^s \rightarrow (-s d_i k)^{-1}$, for $\Lambda = -s$. Therefore \mathcal{Z}_Λ^s can be seen as a generalization of the Elsässer variables to Hall MHD. In the rest of the paper, we will use the relation

$$\mathcal{Z}_\Lambda^s = (\xi_\Lambda^s - \xi_\Lambda^{-s}) a_\Lambda^s e^{-i\omega_\Lambda^s t}. \quad (33)$$

where a_Λ^s is the wave amplitude in the interaction representation for which we have, in the linear approximation, $\partial_t a_\Lambda^s = 0$. In particular, that means that weak nonlinearities will modify only slowly in time the Hall MHD wave amplitudes. The coefficient in front of the wave amplitude is introduced in advance to simplify the algebra that we are going to develop.

3 Helical wave turbulence formalism

3.1 Fundamental equations

We decompose the inviscid nonlinear Hall MHD equations (14)–(15) on the complex helicity decomposition introduced in the previous section. Then we project the equations on vector $\mathbf{h}_{-\mathbf{k}}^\Lambda$. We obtain:

$$\partial_t \mathcal{U}_\Lambda - iB_0 k_\parallel \mathcal{B}_\Lambda = \quad (34)$$

$$-\frac{i\epsilon}{2} \int \sum_{\Lambda_p, \Lambda_q} (\mathcal{U}_{\Lambda_p} \mathcal{U}_{\Lambda_q} - \mathcal{B}_{\Lambda_p} \mathcal{B}_{\Lambda_q}) (\mathbf{k} \cdot \mathbf{h}_{\mathbf{p}}^{\Lambda_p}) (\mathbf{h}_{\mathbf{q}}^{\Lambda_q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda}) \delta_{pq,k} d\mathbf{p} d\mathbf{q},$$

and

$$\partial_t \mathcal{B}_\Lambda - iB_0 k_\parallel \mathcal{U}_\Lambda + i\Lambda d_i B_0 k_\parallel k \mathcal{B}_\Lambda = \quad (35)$$

$$-\frac{i\epsilon}{2} \int \sum_{\Lambda_p, \Lambda_q} (\mathcal{U}_{\Lambda_p} \mathcal{B}_{\Lambda_q} - \mathcal{U}_{\Lambda_q} \mathcal{B}_{\Lambda_p}) (\mathbf{k} \cdot \mathbf{h}_{\mathbf{p}}^{\Lambda_p}) (\mathbf{h}_{\mathbf{q}}^{\Lambda_q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda}) \delta_{pq,k} d\mathbf{p} d\mathbf{q}$$

+

$$\frac{i\epsilon}{2} \int k \Lambda d_i \sum_{\Lambda_p, \Lambda_q} \mathcal{B}_{\Lambda_p} \mathcal{B}_{\Lambda_q} [(\mathbf{q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda}) (\mathbf{h}_{\mathbf{p}}^{\Lambda_p} \cdot \mathbf{h}_{\mathbf{q}}^{\Lambda_q}) - (\mathbf{q} \cdot \mathbf{h}_{\mathbf{p}}^{\Lambda_p}) (\mathbf{h}_{\mathbf{q}}^{\Lambda_q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda})] \delta_{pq,k} d\mathbf{p} d\mathbf{q},$$

where $\delta_{pq,k} = \delta(\mathbf{p} + \mathbf{q} - \mathbf{k})$. The delta distributions come from the Fourier transforms of the nonlinear terms. We introduce the generalized Elsässer

variables a_Λ^s in the interaction representation and we find:

$$\partial_t a_\Lambda^s = \frac{i\epsilon}{2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} L_{\substack{s s_p s_q \\ -k p q}}^{\Lambda \Lambda_p \Lambda_q} a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} e^{-i\Omega_{pq,k} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \quad (36)$$

where

$$L_{\substack{s s_p s_q \\ k p q}}^{\Lambda \Lambda_p \Lambda_q} = \frac{1 - \xi_\Lambda^{s^2}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} (\mathbf{q} \cdot \mathbf{h}_\mathbf{k}^\Lambda) (\mathbf{h}_\mathbf{p}^{\Lambda_p} \cdot \mathbf{h}_\mathbf{q}^{\Lambda_q}) + \frac{\xi_\Lambda^{s^2} + \xi_\Lambda^s \xi_{\Lambda_p}^{-s_p} - \xi_\Lambda^s \xi_{\Lambda_q}^{-s_q} - \xi_{\Lambda_p}^{-s_p} \xi_{\Lambda_q}^{-s_q}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} (\mathbf{q} \cdot \mathbf{h}_\mathbf{p}^{\Lambda_p}) (\mathbf{h}_\mathbf{q}^{\Lambda_q} \cdot \mathbf{h}_\mathbf{k}^\Lambda), \quad (37)$$

and

$$\Omega_{pq,k} = \omega_{\Lambda_p}^{s_p} + \omega_{\Lambda_q}^{s_q} - \omega_\Lambda^s. \quad (38)$$

Equation (36) is the wave amplitude equation from which it is possible to extract some information. As expected we see that the nonlinear terms are of order ϵ . This means that weak nonlinearities will modify only slowly in time the Hall MHD wave amplitude. They contain an exponentially oscillating term which is essential for the asymptotic closure. Indeed, wave turbulence deals with variations of spectral densities at very large time, *i.e.* for a nonlinear transfer time much greater than the wave period. As a consequence, most of the nonlinear terms are destroyed by phase mixing and only a few of them, the resonance terms, survive (see *e.g.* Newell et al., 2001). The expression obtained for the fundamental equation (36) is usual in wave turbulence. The main difference between problems is localized in the matrix L which is interpreted as a complex geometrical coefficient. We will see below that the local decomposition allows to get a polar form for such a coefficient which is much easier to manipulate. From equation (36) we see eventually that, contrary to incompressible MHD, there is no exact solutions to the nonlinear problem in incompressible Hall MHD. The origin of such a difference is that in MHD the nonlinear term involves Alfvén waves traveling only in opposite directions whereas in Hall MHD this constrain does not exist (we have a summation over Λ and s). In other words, if one type of wave is not present in incompressible MHD then the nonlinear term cancels whereas in incompressible Hall MHD it is not the case. The conclusion reached by Mahajan and Krishan (2005) is therefore not correct: the condition $\mathcal{U}_\Lambda = -\xi_\Lambda^s \mathcal{B}_\Lambda$ does not correspond to a nonlinear solution of incompressible Hall MHD.

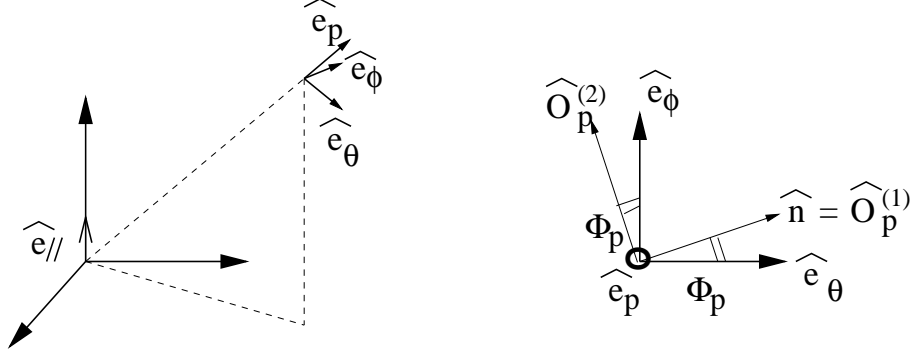


Figure 1: Sketch of the local decomposition for a given wavevector \mathbf{p} .

3.2 Local decomposition

In order to evaluate the scalar products of complex helical vectors found in the geometrical coefficient (37), it is convenient to introduce a vector basis local to each particular triad (Waleffe, 1992; Turner, 2000; Galtier, 2003; Galtier and Bhattacharjee, 2003). For example, for a given vector \mathbf{p} , we define the orthonormal basis vectors,

$$\begin{aligned}\hat{\mathbf{O}}^{(1)}(\mathbf{p}) &= \hat{\mathbf{n}}, \\ \hat{\mathbf{O}}^{(2)}(\mathbf{p}) &= \hat{\mathbf{e}}_p \times \hat{\mathbf{n}}, \\ \hat{\mathbf{O}}^{(3)}(\mathbf{p}) &= \hat{\mathbf{e}}_p,\end{aligned}\tag{39}$$

where $\hat{\mathbf{e}}_p = \mathbf{p}/|\mathbf{p}|$ and

$$\hat{\mathbf{n}} = \frac{\mathbf{p} \times \mathbf{k}}{|\mathbf{p} \times \mathbf{k}|} = \frac{\mathbf{q} \times \mathbf{p}}{|\mathbf{q} \times \mathbf{p}|} = \frac{\mathbf{k} \times \mathbf{q}}{|\mathbf{k} \times \mathbf{q}|}.\tag{40}$$

We see that the vector $\hat{\mathbf{n}}$ is normal to any vector of the triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ and changes sign if \mathbf{p} and \mathbf{q} are interchanged, *i.e.* $\hat{\mathbf{n}}_{(\mathbf{k}, \mathbf{q}, \mathbf{p})} = -\hat{\mathbf{n}}_{(\mathbf{k}, \mathbf{p}, \mathbf{q})}$. Note that $\hat{\mathbf{n}}$ does not change by cyclic permutation, *i.e.* , $\hat{\mathbf{n}}_{(\mathbf{k}, \mathbf{q}, \mathbf{p})} = \hat{\mathbf{n}}_{(\mathbf{q}, \mathbf{p}, \mathbf{k})} = \hat{\mathbf{n}}_{(\mathbf{p}, \mathbf{k}, \mathbf{q})}$. A sketch of the local decomposition is given in Fig. 1. We now introduce the vectors

$$\Xi^{\Lambda_p}(\mathbf{p}) \equiv \Xi_{\mathbf{p}}^{\Lambda_p} = \hat{\mathbf{O}}^{(1)}(\mathbf{p}) + i\Lambda_p \hat{\mathbf{O}}^{(2)}(\mathbf{p}),\tag{41}$$

and define the rotation angle Φ_p , so that

$$\begin{aligned}\cos \Phi_p &= \hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_\theta(\mathbf{p}), \\ \sin \Phi_p &= \hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_\phi(\mathbf{p}).\end{aligned}\tag{42}$$

The decomposition of the helicity vector $\mathbf{h}_p^{\Lambda_p}$ in the local basis gives

$$\mathbf{h}_p^{\Lambda_p} = \Xi_p^{\Lambda_p} e^{i\Lambda_p \Phi_p}. \quad (43)$$

After some algebra we obtain the following polar form for the matrix L ,

$$\begin{aligned} L_{k p q}^{\Lambda_p \Lambda_q s p s q} = & \\ i e^{i(\Lambda \Phi_k + \Lambda_p \Phi_p + \Lambda_q \Phi_q)} \frac{\Lambda \Lambda_p \Lambda_q}{\xi_\Lambda^s - \xi_\Lambda^{-s}} \frac{\sin \psi_k}{k} [(\xi_\Lambda^{s^2} - \xi_{\Lambda_p}^{-s p} \xi_{\Lambda_q}^{-s q}) \Lambda_q q (\Lambda k + \Lambda_p p + \Lambda_q q) & \\ + & \\ \xi_\Lambda^s (\xi_{\Lambda_p}^{-s p} - \xi_{\Lambda_q}^{-s q}) k q (\Lambda \Lambda_q + \cos \psi_p)] . & \end{aligned} \quad (44)$$

The angle ψ_k refers to the angle opposite to \mathbf{k} in the triangle defined by $\mathbf{k} = \mathbf{p} + \mathbf{q}$ ($\sin \psi_k = \hat{\mathbf{n}} \cdot (\mathbf{q} \times \mathbf{p}) / |(\mathbf{q} \times \mathbf{p})|$). To obtain equation (44), we have also used the well-known triangle relations

$$\frac{\sin \psi_k}{k} = \frac{\sin \psi_p}{p} = \frac{\sin \psi_q}{q}. \quad (45)$$

Further modifications have to be made before applying the spectral formalism. In particular, the fundamental equation has to be invariant under interchange of \mathbf{p} and \mathbf{q} . To do so, we introduce the symmetrized and renormalized matrix M :

$$M_{k p q}^{\Lambda_p \Lambda_q s p s q} = i d_i \frac{1}{\xi_\Lambda^{s^2}} \frac{\xi_\Lambda^{s q} - \xi_\Lambda^{-s}}{\xi_{\Lambda_q}^{s q} - \xi_{\Lambda_p}^{s p}} \left(L_{k p q}^{\Lambda_p \Lambda_q s p s q} + L_{k q p}^{\Lambda_q \Lambda_p s q s p} \right). \quad (46)$$

Finally, by using the identities given in Appendix A, we obtain:

$$\partial_t a_\Lambda^s = \frac{\epsilon}{4 d_i} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s p, s q}} \xi_\Lambda^{s^2} \frac{\xi_{\Lambda_q}^{s q} - \xi_{\Lambda_p}^{s p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} M_{k p q}^{\Lambda_p \Lambda_q s p s q} a_{\Lambda_p}^{s p} a_{\Lambda_q}^{s q} e^{-i\Omega_{pq,k} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \quad (47)$$

where

$$\begin{aligned} M_{k p q}^{\Lambda_p \Lambda_q s p s q} = & \\ e^{i(\Lambda \Phi_k + \Lambda_p \Phi_p + \Lambda_q \Phi_q)} \Lambda \Lambda_p \Lambda_q \frac{\sin \psi_k}{k} (\Lambda k + \Lambda_p p + \Lambda_q q) (1 - \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s p^2} \xi_{\Lambda_q}^{-s q^2}). & \end{aligned} \quad (48)$$

The matrix M possesses the following properties ($*$ denotes the complex conjugate),

$$\left(M \begin{smallmatrix} \Lambda\Lambda_p\Lambda_q \\ s\ s_p\ s_q \\ k\ p\ q \end{smallmatrix} \right)^* = M \begin{smallmatrix} -\Lambda-\Lambda_p-\Lambda_q \\ -s-s_p-s_q \\ k\ p\ q \end{smallmatrix} = M \begin{smallmatrix} \Lambda\ \Lambda_p\ \Lambda_q \\ s\ s_p\ s_q \\ -k\ -p\ -q \end{smallmatrix}, \quad (49)$$

$$M \begin{smallmatrix} \Lambda\Lambda_p\Lambda_q \\ s\ s_p\ s_q \\ k\ p\ q \end{smallmatrix} = -M \begin{smallmatrix} \Lambda\Lambda_q\Lambda_p \\ s\ s_q\ s_p \\ k\ q\ p \end{smallmatrix}, \quad (50)$$

$$M \begin{smallmatrix} \Lambda\Lambda_p\Lambda_q \\ s\ s_p\ s_q \\ k\ p\ q \end{smallmatrix} = -M \begin{smallmatrix} \Lambda_q\Lambda_p\Lambda \\ s_q\ s_p\ s \\ q\ p\ k \end{smallmatrix}, \quad (51)$$

$$M \begin{smallmatrix} \Lambda\Lambda_p\Lambda_q \\ s\ s_p\ s_q \\ k\ p\ q \end{smallmatrix} = -M \begin{smallmatrix} \Lambda_p\Lambda\Lambda_q \\ s_p\ s\ s_q \\ p\ k\ q \end{smallmatrix}. \quad (52)$$

Equation (47) is the fundamental equations that describe the slow evolution of the Alfvén wave amplitudes due to the nonlinear terms of the incompressible Hall MHD equations. It is the starting point for deriving the wave kinetic equations. The local decomposition used here allows us to represent concisely complex information in an exponential function. As we will see, it will simplify significantly the derivation of the wave kinetic equations.

From equation (47) we note that the nonlinear coupling between helicity states associated with wavevectors, \mathbf{p} and \mathbf{q} , vanishes when the wavevectors are collinear (since then, $\sin \psi_k = 0$). This property is similar to the one found in the limit of EMHD (Galtier and Bhattacharjee, 2003). It seems to be a general property for helicity waves (Kraichnan, 1973; Waleffe, 1993; Turner, 2000; Galtier, 2003). Additionally, we note that the nonlinear coupling between helicity states vanishes whenever the wavenumbers p and q are equal if their associated wave and directional polarities, Λ_p , Λ_q , and s_p , s_q respectively, are also equal. In the case of whistler (EMHD) waves, for which we have $\Lambda = s$ (right circularly polarized), this property was already observed (Galtier and Bhattacharjee, 2003). Here we generalize this finding to right and left circularly polarized waves. In the large scale limit, *i.e.* when we tend to pure incompressible MHD, this property tends to disappear. For pure incompressible MHD, where Alfvén waves are linearly polarized, this is not observed anymore. As noticed before, the nature of the polarization seems to be fundamental.

We are interested by the long-time behavior of the Alfvén wave amplitudes. From the fundamental equation (47), we see that the nonlinear wave

coupling will come from resonant terms such that,

$$\begin{cases} \mathbf{k} = \mathbf{p} + \mathbf{q}, \\ k_{\parallel} \xi_{\Lambda}^s = p_{\parallel} \xi_{\Lambda_p}^{s_p} + q_{\parallel} \xi_{\Lambda_q}^{s_q}. \end{cases} \quad (53)$$

The resonance condition may also be written:

$$\frac{\xi_{\Lambda}^s - \xi_{\Lambda_p}^{s_p}}{q_{\parallel}} = \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda}^s}{p_{\parallel}} = \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_{\parallel}}. \quad (54)$$

As we will see below, the relations (54) are useful in simplifying the wave kinetic equations and demonstrating the conservation of ideal invariants. In particular, we note that we recover the resonance conditions for whistler waves by taking the appropriate limit.

3.3 Dynamics and wave kinetic equations

Fully developed wave turbulence is a state of a system composed of many simultaneously excited and interacting nonlinear waves where the energy distribution, far from thermodynamic equilibrium, is characterized by a wide power law spectrum. This range of wavenumbers, the inertial range, is generally localized between large scales at which energy is injected in the system and small dissipative scales. The origin of wave turbulence dates back to the early sixties and since then many papers have been devoted to the subject (see *e.g.* Hasselmann, 1962; Benney and Saffman, 1966; Zakharov, 1967; Benney and Newell, 1969; Sagdeev and Galeev, 1969; Kuznetsov, 1972; Zakharov et al., 1992; Newell et al., 2001). The essence of weak wave turbulence is the statistical study of large ensembles of weakly interacting dispersive waves via a systematic asymptotic expansion in powers of small nonlinearity. This technique leads finally to the derivation of wave kinetic equations for quantities like the energy and more generally for the quadratic invariants of the system studied. Here, we will follow the standard Eulerian formalism of wave turbulence (see *e.g.* Benney and Newell, 1969).

We define the density tensor $q_{\Lambda}^s(\mathbf{k})$ for an homogeneous turbulence, such that:

$$\langle a_{\Lambda}^s(\mathbf{k}) a_{\Lambda'}^{s'}(\mathbf{k}') \rangle \equiv q_{\Lambda}^s(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta_{\Lambda\Lambda'} \delta_{ss'}, \quad (55)$$

for which we shall write a “closure” equation. The presence of the delta $\delta_{\Lambda\Lambda'}$ and $\delta_{ss'}$ means that correlations with opposite wave or directional polarities

have no long-time influence in the wave turbulence regime; the third delta distribution $\delta(\mathbf{k} + \mathbf{k}')$ is the consequence of the homogeneity assumption. It is strongly linked to the form of the frequency ω_Λ^s (see the discussion in Section 4.5). Details of the derivation of the wave kinetic equations are given in Appendix B. We obtain the following result:

$$\begin{aligned} \partial_t q_\Lambda^s(\mathbf{k}) = & \quad (56) \\ & \frac{\pi \epsilon^2}{4 d_i^2 B_0^2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2 \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_\parallel} \right)^2 \\ & \left(\frac{\omega_\Lambda^s}{1 + \xi_\Lambda^{-s^2}} \right) \left[\left(\frac{\omega_\Lambda^s}{1 + \xi_\Lambda^{-s^2}} \right) \frac{1}{q_\Lambda^s(\mathbf{k})} - \left(\frac{\omega_{\Lambda_p}^{s_p}}{1 + \xi_{\Lambda_p}^{-s_p^2}} \right) \frac{1}{q_{\Lambda_p}^{s_p}(\mathbf{p})} - \left(\frac{\omega_{\Lambda_q}^{s_q}}{1 + \xi_{\Lambda_q}^{-s_q^2}} \right) \frac{1}{q_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \\ & q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

Equation (56) is the main result of the helical wave turbulence formalism. It describes the statistical properties of incompressible Hall MHD wave turbulence at the lowest order, *i.e.* for three-wave interactions.

4 General properties of Hall MHD wave turbulence

4.1 Triadic conservation of ideal invariants

In section 2.3, we have introduced the 3D ideal invariants of incompressible Hall MHD. The first test that the wave kinetic equations have to satisfy is the detailed conservation of these invariants, that is to say, the conservation of invariants for each triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$. Starting from definitions (6)–(8), we note that the Fourier spectra of the ideal invariants are:

$$E(\mathbf{k}) = \sum_{\Lambda, s} (1 + \xi_\Lambda^{-s^2}) q_\Lambda^s(\mathbf{k}), \quad (57)$$

$$H_m(\mathbf{k}) = \sum_{\Lambda, s} \frac{\Lambda}{k} q_\Lambda^s(\mathbf{k}), \quad (58)$$

$$H_G(\mathbf{k}) = \sum_{\Lambda, s} \frac{\Lambda \xi_\Lambda^{-s^4}}{k} q_\Lambda^s(\mathbf{k}). \quad (59)$$

We will first check the energy conservation. From expression (56), we find:

$$\begin{aligned} \partial_t E(t) &\equiv \partial_t \int E(\mathbf{k}) d\mathbf{k} \equiv \partial_t \int \sum_{\Lambda, s} e_{\Lambda}^s(\mathbf{k}) d\mathbf{k} = \\ &\frac{\pi \epsilon^2}{4 d_i^2 B_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2 \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_{\parallel}} \right)^2 \\ &\omega_{\Lambda}^s \left[\frac{\omega_{\Lambda}^s}{e_{\Lambda}^s(\mathbf{k})} - \frac{\omega_{\Lambda_p}^{s_p}}{e_{\Lambda_p}^{s_p}(\mathbf{p})} - \frac{\omega_{\Lambda_q}^{s_q}}{e_{\Lambda_q}^{s_q}(\mathbf{q})} \right] q_{\Lambda}^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{k} d\mathbf{p} d\mathbf{q}. \end{aligned} \quad (60)$$

Equation (60) is invariant under cyclic permutations of wavevectors. That leads to:

$$\begin{aligned} \partial_t E(t) &= \\ &\frac{\pi \epsilon^2}{12 d_i^2 B_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2 \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_{\parallel}} \right)^2 \\ &\Omega_{k,pq} \left[\frac{\omega_{\Lambda}^s}{e_{\Lambda}^s(\mathbf{k})} - \frac{\omega_{\Lambda_p}^{s_p}}{e_{\Lambda_p}^{s_p}(\mathbf{p})} - \frac{\omega_{\Lambda_q}^{s_q}}{e_{\Lambda_q}^{s_q}(\mathbf{q})} \right] q_{\Lambda}^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{k} d\mathbf{p} d\mathbf{q}. \end{aligned} \quad (61)$$

Total energy is conserved exactly on the resonant manifold since then $\Omega_{k,pq} = 0$: we have triadic conservation of total energy.

For the other invariants, it is straightforward to show that the difference between them is conserved by the wave kinetic equations. With relation (128), we obtain:

$$\begin{aligned} \partial_t (H_m(t) - H_G(t)) &\equiv \partial_t \int (H_m(\mathbf{k}) - H_G(\mathbf{k})) d\mathbf{k} = \\ &\frac{\pi \epsilon^2}{4 d_i B_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2 \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_{\parallel}} \right)^2 \\ &k_{\parallel} \left[\left(\frac{\omega_{\Lambda}^s}{1 + \xi_{\Lambda}^{-s^2}} \right) \frac{1}{q_{\Lambda}^s(\mathbf{k})} - \left(\frac{\omega_{\Lambda_p}^{s_p}}{1 + \xi_{\Lambda_p}^{-s_p^2}} \right) \frac{1}{q_{\Lambda_p}^{s_p}(\mathbf{p})} - \left(\frac{\omega_{\Lambda_q}^{s_q}}{1 + \xi_{\Lambda_q}^{-s_q^2}} \right) \frac{1}{q_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \\ &q_{\Lambda}^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{k} d\mathbf{p} d\mathbf{q}. \end{aligned} \quad (62)$$

Equation (60) is also invariant under cyclic permutations of wavevectors. Then one is led to

$$\partial_t (H_m(t) - H_G(t)) = \quad (63)$$

$$\begin{aligned}
& \frac{\pi \epsilon^2}{12 d_i B_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2 \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_\parallel} \right)^2 \\
& (k_\parallel - p_\parallel - q_\parallel) \left[\left(\frac{\omega_\Lambda^s}{1 + \xi_\Lambda^{-s^2}} \right) \frac{1}{q_\Lambda^s(\mathbf{k})} - \left(\frac{\omega_{\Lambda_p}^{s_p}}{1 + \xi_{\Lambda_p}^{-s_p^2}} \right) \frac{1}{q_{\Lambda_p}^{s_p}(\mathbf{p})} - \left(\frac{\omega_{\Lambda_q}^{s_q}}{1 + \xi_{\Lambda_q}^{-s_q^2}} \right) \frac{1}{q_{\Lambda_q}^{s_q}(\mathbf{q})} \right] \\
& q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{k} d\mathbf{p} d\mathbf{q},
\end{aligned}$$

which is exactly equal to zero on the resonant manifold: we have triadic conservation of the difference between magnetic and global helicity. The last sufficient step would be to show detailed conservation for one of the two helicities. Unfortunately it is not so trivial and we will not give the proof here. However, we note already that magnetic helicity is conserved for equilateral triangles ($k = p = q$), *i.e.* for strongly local interactions.

4.2 General properties

From the wave kinetic equations (56), we find several general properties. Some of them can be obtained directly from the wave amplitude equation (47) as explained in Section 3.2. First, we observe that there is no coupling between Hall MHD waves associated with wavevectors, \mathbf{p} and \mathbf{q} , when the wavevectors are collinear ($\sin \psi_k = 0$). Second, we note that there is no coupling between helical waves associate with these vectors whenever the magnitudes, p and q , are equal if their associated polarities, s_p and s_q in one hand and, Λ_p and Λ_q on the other hand, are also equal (since then, $\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p} = 0$). These properties hold for the three inviscid invariants and generalize what was found previously in EMHD (Galtier and Bhattacharjee, 2003) where we only have right circularly polarized waves ($\Lambda = s$). It seems to be a generic property of helical wave interactions (Kraichnan, 1973; Waleffe, 1993; Turner, 2000; Galtier, 2003). As noted before, this property tends to disappear when the large scale limit is taken, *i.e.* when we tend to standard MHD. Third, it follows from the previous observations that a strong helical perturbation localized initially in a narrow band of wavenumbers will lead to a weak transfer of energy, magnetic and global helicities. Note that these properties can be inferred from the fundamental equation (47) as well.

4.3 High frequency limit of electron MHD

In the present section we shall demonstrate that the small scale limit ($d_i k \rightarrow +\infty$) of the wave kinetic equations (56) tends to the expected equations of electron MHD when only right ($\Lambda = s$) circularly polarized waves are taken into account. We recall that, in the small scale limit, $\xi_s^s \rightarrow -s d_i k$ and $\xi_s^{-s} \rightarrow s/d_i k$. One obtains

$$\begin{aligned} \partial_t q_s^s(\mathbf{k}) = & \quad (64) \\ & \frac{\pi d_i^2 \epsilon^2}{4} \int \sum_{s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 (sk + s_p p + s_q q)^2 \left(\frac{s_q q - s_p p}{k_{\parallel}} \right)^2 s k k_{\parallel} \\ & \left[s k k_{\parallel} q_{s_p}^{s_p}(\mathbf{p}) q_{s_q}^{s_q}(\mathbf{q}) - s_p p p_{\parallel} q_s^s(\mathbf{k}) q_{s_q}^{s_q}(\mathbf{q}) - s_q q q_{\parallel} q_s^s(\mathbf{k}) q_{s_p}^{s_p}(\mathbf{p}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}, \end{aligned}$$

where $\Omega_{k,pq} = B_0 d_i (sk_{\parallel} k - s_p p_{\parallel} p - s_q q_{\parallel} q)$. The kinetic equations found have exactly the same form as in Galtier and Bhattacharjee (2003) (where by definition $\omega_s^s = s \omega_k$, $B_0 = 1$ and $d_i = 1$) who derived a wave turbulence theory for electron MHD (see the discussion in Section 5). In particular, this means that by recovering the electron MHD description from the present Hall MHD theory, we recover all the properties already found for whistler wave turbulence (anisotropy, scaling laws, direct cascade...). We will come back to that point in Section 5. Note finally that there is a strong analogy between such a limit and wave turbulence in rapidly rotating flows (Galtier, 2003; Bellet et al., 2005; Morize et al., 2005). The physical reason is that in both problems (i) there is a privileged direction, played by the rotating axis or the magnetic field \mathbf{B}_0 , (ii) there are dispersive helical waves, called inertial waves for rotating flows, and (iii) the wave frequencies are not very different, *i.e.* the inertial wave frequency is proportional to k_{\parallel}/k . In the past, comparisons have been made between incompressible rotating turbulence and magnetized plasmas described by incompressible MHD. The results obtained in the framework of wave turbulence show clearly that the comparison is much more relevant if one considers electron MHD plasmas. Indeed, a small nonlinear transfer is found along the privileged direction for rotating and electron MHD turbulence whereas this kind of transfer is strictly forbidden in MHD turbulence.

4.4 High frequency limit of ion MHD

We may also be interested by the small scale limit ($d_i k \rightarrow +\infty$) of the wave kinetic equations (56) when only left ($\Lambda = -s$) circularly polarized waves

are taken into account. We decide to call this limit the ion MHD (IMHD) approximation following the example of the electron MHD. We have, for such a limit, $\xi_{-s}^{-s} \rightarrow s d_i k$ and $\xi_{-s}^s \rightarrow -s/d_i k$. One finds

$$\begin{aligned} \partial_t q_{-s}^s(\mathbf{k}) = & \quad (65) \\ & \frac{\pi d_i^2 \epsilon^2}{4} \int \sum_{s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 (s k + s_p p + s_q q)^2 \left(\frac{s_q q - s_p p}{p q k_{\parallel}} \right)^2 s k_{\parallel} k p^4 q^4 \delta(\Omega_{k,pq}) \delta_{k,pq} \\ & \left[\frac{s k_{\parallel}}{k^3} q_{-s_p}^{s_p}(\mathbf{p}) q_{-s_q}^{s_q}(\mathbf{q}) - \frac{s_p p_{\parallel}}{p^3} q_{-s}^s(\mathbf{k}) q_{-s_q}^{s_q}(\mathbf{q}) - \frac{s_q q_{\parallel}}{q^3} q_{-s}^s(\mathbf{k}) q_{-s_p}^{s_p}(\mathbf{p}) \right] d\mathbf{p} d\mathbf{q}, \end{aligned}$$

with $\Omega_{k,pq} = (B_0/d_i)(s k_{\parallel}/k - s_p p_{\parallel}/p - s_q q_{\parallel}/q)$. No existing theory has been developed for such a limit. We will see in Section 5 that it is possible to extract from the master equations of ion cyclotron turbulence some exact properties.

4.5 Low frequency limit of standard MHD

This section is devoted to the low frequency limit, *i.e.* the large scale limit, of the wave kinetic equations (56) of Hall MHD. The MHD limit is somewhat singular for Hall MHD. Indeed the large scale limit does not tend to the expected wave kinetic equations that were derived first by Galtier et al. (2000). The subtle point resides in the kinematics: for pseudo-dispersive MHD waves, the definition (55) of the density tensor $q_{\Lambda}^s(\mathbf{k})$ that is used for Hall MHD is not valid anymore. The reason is that in the large scale limit the polarization Λ does not appear anymore in the frequency, which is $\omega = s k_{\parallel} B_0$. Then the kinematics tells us that the definition for the density tensor is

$$\langle a_{\Lambda}^s(\mathbf{k}) a_{\Lambda'}^{s'}(\mathbf{k}') \rangle \equiv \tilde{q}_{\Lambda\Lambda'}^{ss'} \delta(\mathbf{k} + \mathbf{k}') \delta_{ss'}, \quad (66)$$

where the condition $\Lambda = \Lambda'$ is not necessary satisfied. This means that the large scale limit of equations (56) leads to MHD wave kinetic equations in the particular case where helicity terms are supposed to be absent and where equality between shear-Alfvén and pseudo-Alfvén waves energy is assumed. Indeed, helicity terms involve quantities for which $\Lambda = -\Lambda'$ and the total energy involves only terms for which $\Lambda = \Lambda'$. However, energies for shear-Alfvén waves and pseudo-Alfvén waves involve terms with different polarities. For shear-Alfvén waves, we have:

$$\mathbf{v}_{\mathbf{k}} - s \mathbf{b}_{\mathbf{k}} = \mathcal{Z}_+^s \mathbf{h}_{\mathbf{k}}^+ + \mathcal{Z}_-^s \mathbf{h}_{\mathbf{k}}^- = (\mathcal{Z}_+^s - \mathcal{Z}_-^s) i \hat{\mathbf{e}}_{\Phi}, \quad (67)$$

and for pseudo-Alfvén waves:

$$\mathbf{v}_{\mathbf{k}} - s\mathbf{b}_{\mathbf{k}} = \mathcal{Z}_+^s \mathbf{h}_{\mathbf{k}}^+ + \mathcal{Z}_-^s \mathbf{h}_{\mathbf{k}}^- = (\mathcal{Z}_+^s + \mathcal{Z}_-^s) \hat{\mathbf{e}}_\theta. \quad (68)$$

Energies associated to shear- and pseudo-Alfvén waves are respectively:

$$(\mathcal{Z}_+^s - \mathcal{Z}_-^s)(\mathcal{Z}_+^s - \mathcal{Z}_-^s)^* = |\mathcal{Z}_+^s|^2 + |\mathcal{Z}_-^s|^2 - \mathcal{Z}_+^s(\mathcal{Z}_-^s)^* - \mathcal{Z}_-^s(\mathcal{Z}_+^s)^*, \quad (69)$$

and

$$(\mathcal{Z}_+^s + \mathcal{Z}_-^s)(\mathcal{Z}_+^s + \mathcal{Z}_-^s)^* = |\mathcal{Z}_+^s|^2 + |\mathcal{Z}_-^s|^2 + \mathcal{Z}_+^s(\mathcal{Z}_-^s)^* + \mathcal{Z}_-^s(\mathcal{Z}_+^s)^*. \quad (70)$$

We see clearly that if, in the wave kinetic equations, we only take into account (quadratic) terms with the same polarizations (Λ) then it is equivalent to assume equality between shear- and pseudo-Alfvén wave energies. We will see below that the large scale limit of the wave kinetic equations (56) of Hall MHD tends to the expected MHD counterpart when the previous assumptions about helicities and energies are satisfied. A derivation of the wave kinetic equations is given in Appendix C. The result is given for the density tensor $q_{\Lambda\Lambda'}^{ss'}$. Contrary to Hall MHD, and actually to any problem in wave turbulence, principal value terms appear for incompressible MHD. The reason of the presence of principal value terms is linked to the nature of Alfvén waves which are pseudo-dispersive.

Further comparisons between the results in Appendix C and the wave kinetic equations obtained by Galtier et al. (2000) are of course possible (see *e.g.* the discussion in Section 5) but, for simplicity, we prefer to focus our attention to the case where the density tensor is symmetric in Λ . Therefore we start our analysis with the general kinetic equation (56) and take the large scale limit ($d_i k \rightarrow 0$) for which we have, at the leading order, $\xi_\Lambda^s \rightarrow -s - \Lambda d_i k/2$. After some simplifications, we arrive at:

$$\partial_t q_\Lambda^s(\mathbf{k}) = \quad (71)$$

$$\frac{\pi \epsilon^2}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (s k \Lambda + s_p p \Lambda_p + s_q q \Lambda_q)^2 \left(\frac{s_q - s_p}{k_\parallel} \right)^2$$

$$s k_\parallel \left(s k_\parallel q_{\Lambda_p}^{s_p}(\mathbf{p}) q_{\Lambda_q}^{s_q}(\mathbf{q}) - s_p p_\parallel q_\Lambda^s(\mathbf{k}) q_{\Lambda_q}^{s_q}(\mathbf{q}) - s_q q_\parallel q_\Lambda^s(\mathbf{k}) q_{\Lambda_p}^{s_p}(\mathbf{p}) \right) \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q},$$

where $\Omega_{k,pq} = B_0(s k_\parallel - s_p p_\parallel - s_q q_\parallel)$. Note that we only have a nonlinear contribution when the wave polarities s_p and s_q are different. We recover

here a well-known property of incompressible MHD: in such a limit, we only have nonlinear interactions between Alfvén waves propagating in different directions. One expands the summation over the directional polarities s_p and s_q , and finds

$$\begin{aligned} \partial_t q_\Lambda^s(\mathbf{k}) = & \quad (72) \\ \frac{\pi \epsilon^2}{4B_0} \int \sum_{\Lambda_p, \Lambda_q} \left(\frac{\sin \psi_k}{k} \right)^2 & (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (\Lambda k + \Lambda_p p - \Lambda_q q)^2 \\ q_{\Lambda_q}^{-s}(\mathbf{q}) \left[q_{\Lambda_p}^s(\mathbf{p}) - q_\Lambda^s(\mathbf{k}) \right] & \delta(q_\parallel) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

This result is exactly the same as in Appendix D when MHD wave kinetic equations are considered in the particular case where only terms symmetric in Λ are retained, *i.e.* terms like $\tilde{q}_{\Lambda\Lambda}^{ss}$. Therefore under these assumptions the MHD description does not appear like a singular limit for Hall MHD. (Note that this continuity in the description is *a priori* not so trivial if we remember that as soon as the Hall term is included in the standard MHD equations, whatever its magnitude is, it changes the polarity of the Alfvén waves which become circularly polarized.)

The comparison with the wave kinetic equations derived by Galtier et al. (2000) (see equations (26)) is not direct since here the problem has been decomposed at the origin on a complex helicity basis. The main signature of this decomposition is the dependency on the wave polarity Λ . In spite of this difficulty, a common point is clearly seen in the presence of the delta $\delta(q_\parallel)$ which arises because of the three-wave frequency resonance condition. This means that in any triadic resonant interaction, there is always one wave that corresponds to a purely 2D motion ($q_\parallel = 0$) whereas the two others have equal parallel components ($p_\parallel = k_\parallel$). The direct consequence is the absence of nonlinear transfer along \mathbf{B}_0 , a result predicted earlier by several authors (see *e.g.* Montgomery and Turner, 1981; Shebalin et al., 1983). In other words, we have a two-dimensionalization of the Alfvén wave turbulence (see also *e.g.* Ng and Bhattacharjee, 1997; Lithwick and Goldreich, 2003).

5 Master equations of Hall MHD turbulence

5.1 General case

In order to extract further information about Hall MHD turbulence, we are going to write the expression of the spectral density $q_\Lambda^s(\mathbf{k})$ in terms of the

invariants. In practice, we need to add a fourth variable, E_d , which has been chosen to be the difference between the kinetic and magnetic energy, namely

$$E_d(\mathbf{k}) = \sum_{\Lambda, s} (\xi_{\Lambda}^{-s^2} - 1) q_{\Lambda}^s(\mathbf{k}). \quad (73)$$

Note that contrary to the MHD case, wave turbulence in Hall MHD allows to have a departure from equipartition between kinetic and magnetic energy and therefore a non trivial value for E_d . We will see, in Section 7, that this property is fundamental to get non trivial nonlocal interactions between waves. It is possible to inverse the system $q_{\Lambda}^s(E, E_d, H_m, H_G)$; one finds

$$q_{\Lambda}^s(\mathbf{k}) = \quad (74)$$

$$\frac{(\xi_{\Lambda}^{s^2} + \xi_{\Lambda}^{-s^2})[(\xi_{\Lambda}^{s^2} - 1)E(\mathbf{k}) - (\xi_{\Lambda}^{s^2} + 1)E_d(\mathbf{k})] + 2\Lambda k(\xi_{\Lambda}^{s^4}H_m(\mathbf{k}) - H_G(\mathbf{k}))}{4(\xi_{\Lambda}^{s^4} - \xi_{\Lambda}^{-s^4})}.$$

The introduction of expression (74) into (56) leads to the wave kinetic equations for E , E_d , H_m and H_G . We will not make such a lengthy development and we will rather focus on the master equations which drive the Hall MHD turbulence. Indeed, because of the presence of the factor Λ in expression (74), we see that the nonlinear terms with different polarities will not play the same role. In particular, for the wave kinetic equations of energies, only the interactions between nonlinear terms involving either energies or helicities will give a contribution. For the wave kinetic equations of helicities, we only have contributions from nonlinear terms involving energies and helicities. Thus the energy equations are the master equations driving Hall MHD turbulence. In other words, this means that an initial state with zero helicity will not generate any helicity at any scale. However, an initial state of zero helicity does not preclude the development of energy spectra. Assuming $H_m(\mathbf{k}) = 0$ and $H_G(\mathbf{k}) = 0$, the master kinetic equations of Hall MHD turbulence, written for the kinetic energy E^V and the magnetic energy E^B , are

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \quad (75)$$

$$\frac{\pi \epsilon^2}{8 d_i^2 B_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 \frac{(\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2}{(1 + \xi_{\Lambda}^{-s^2})(1 + \xi_{\Lambda_p}^{-s_p^2})(1 + \xi_{\Lambda_q}^{-s_q^2})}$$

$$\left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_{\parallel}} \right)^2 \left\{ \begin{matrix} \xi_{\Lambda}^{-s^2} \\ 1 \end{matrix} \right\} \frac{\omega_{\Lambda}^s \omega_{\Lambda_p}^{s_p}}{\xi_{\Lambda}^{-s^2} + 1} \left(\frac{\xi_{\Lambda_q}^{-s_q^2} E^V(\mathbf{q}) - E^B(\mathbf{q})}{\xi_{\Lambda_q}^{-s_q^2} - 1} \right) \left[\left(\frac{\xi_{\Lambda_p}^{-s_p^2} E^V(\mathbf{p}) - E^B(\mathbf{p})}{\xi_{\Lambda_p}^{-s_p^2} - 1} \right) - \left(\frac{\xi_{\Lambda}^{-s^2} E^V(\mathbf{k}) - E^B(\mathbf{k})}{\xi_{\Lambda}^{-s^2} - 1} \right) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

In Appendix E the equivalent kinetic equations for the variables E and E_d are given. Below we describe the small scale (whistler and ion cyclotron waves) and large scale (pure Alfvén waves) limits of such a description. In particular, we will see that the role played by the kinetic and magnetic energies may be very different and that the master equations can be simplified further.

5.2 Master equations in the limit of Whistler wave turbulence

In the small scale limit ($d_i k \rightarrow +\infty$), one can distinguish between the whistler branch and the ion cyclotron branch. For right circularly polarized wave ($\Lambda = s$) we have $\xi_{\Lambda}^{-s^2} \rightarrow 1/d_i^2 k^2$, whereas for left circularly polarized wave ($\Lambda = -s$), $\xi_{\Lambda}^{-s^2} \rightarrow d_i^2 k^2$. Then in the limit of whistler wave turbulence (EMHD limit), we obtain

$$\partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = \quad (76)$$

$$\frac{\pi \epsilon^2 d_i^2}{8} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 (sk + s_p p + s_q q)^2 \left(\frac{s_q q - s_p p}{k_{\parallel}} \right)^2 \left\{ \begin{matrix} 1/d_i^2 k^2 \\ 1 \end{matrix} \right\} s k_{\parallel} k s_p p_{\parallel} p E^B(\mathbf{q}) [E^B(\mathbf{p}) - E^B(\mathbf{k})] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q},$$

with $\Omega_{k,pq} = B_0 d_i (sk_{\parallel} k - s_p p_{\parallel} p - s_q q_{\parallel} q)$. We see clearly that the kinetic energy is not a relevant quantity in such a limit since the factor $1/d_i^2 k^2$ damps the nonlinear contributions. This is not in contradiction with what we know if we remember that the velocity in the original Hall MHD equations is a combination of the ion and electron velocities. In the EMHD limit, *i.e.* in the small spatio-temporal scale limit, this velocity tends to be small: ions do not have time to follow electrons and they provide a static homogeneous background on which electrons move. Since the magnetic field is frozen into the ideal electron flow (see Section 2.3 for the proof) we understand

why the magnetic energy is the dominant contribution to the wave kinetic equations. The wave kinetic equation found for the magnetic energy have, at leading order, exactly the same form as in Galtier and Bhattacharjee (2003) (where $\omega_s^s = s \omega_k$ and $B_0 = 1$) but they are more general in the sense that we have at our disposal not only an equation for the magnetic field, the dominant contribution, but also for the velocity that behaves like the magnetic field. The consequence is that we recover all the properties already known. In particular, in such a regime the energy spectrum follows the power law $E(k_\perp, k_\parallel) \sim k_\perp^{-5/2} k_\parallel^{-1/2}$ (k_\parallel is assumed positive) which is an exact solution of the wave kinetic equations.

5.3 Master equations in the limit of ion cyclotron wave turbulence

For the ion cyclotron branch ($\Lambda = -s$), the small scale limit ($d_i k \rightarrow +\infty$) gives

$$\begin{aligned} \partial_t \left\{ \begin{array}{c} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{array} \right\} = & \quad (77) \\ \frac{\pi \epsilon^2}{8 d_i^2} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 (sk + s_p p + s_q q)^2 \left(\frac{s_p p - s_q q}{k_\parallel p q} \right)^2 & \\ \left\{ \begin{array}{c} d_i^2 k^2 \\ 1 \end{array} \right\} \frac{sk_\parallel s_p p_\parallel p q^2}{k} E^V(\mathbf{q}) [E^V(\mathbf{p}) - E^V(\mathbf{k})] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}, & \end{aligned}$$

with $\Omega_{k,pq} = (B_0/d_i)(sk_\parallel/k - s_p p_\parallel/p - s_q q_\parallel/q)$. In such a limit, we are able to describe both the kinetic and the magnetic energies but we see that the dominant quantity is the kinetic energy since the factor $d_i^2 k^2$ amplifies the nonlinear contributions in the corresponding equation (see *e.g.* Mahajan and Krishan, 2005). The physical reason is that in such a limit, we are mainly dealing with ions whereas the magnetic field is frozen into the ideal electron flow (see Section 2.3). Like for the EMHD limit, it is possible to derive the exact power law solutions. First of all, we note that for strongly local interactions the resonance condition writes

$$\left(\frac{s_p - s_q}{k_\parallel} \right)^2 \approx \left(\frac{s_q - s}{p_\parallel} \right)^2 \approx \left(\frac{s - s_p}{q_\parallel} \right)^2, \quad (78)$$

which leads, as for the whistler waves case, to anisotropic turbulence. The same analysis for strongly nonlocal interactions leads to the same conclusion.

Thus, we may write the wave kinetic equations in the limit $k_\perp \gg k_\parallel$. For an axisymmetric turbulence $E^V(\mathbf{k}_\perp, k_\parallel) = E^V(k_\perp, k_\parallel)/2\pi k_\perp$; we find

$$\partial_t E^V(k_\perp, k_\parallel) = \quad (79)$$

$$\frac{\epsilon^2}{16} \int \sum_{s, s_p, s_q} \sin \psi_{q_\perp} (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel p_\perp q_\perp} \right)^2 s k_\parallel s_p p_\parallel$$

$$E^V(q_\perp, q_\parallel) \left[k_\perp E^V(p_\perp, p_\parallel) - p_\perp E^V(k_\perp, k_\parallel) \right] \delta(\Omega_{k,pq}) \delta_{k_\parallel, p_\parallel q_\parallel} dp_\perp dq_\perp dp_\parallel dq_\parallel,$$

where $\Omega_{k,pq} = (B_0/d_i)(s k_\parallel/k_\perp - s_p p_\parallel/p_\perp - s_q q_\parallel/q_\perp)$. Note that in such a limit the well-known triangle relations are: $\sin \psi_{k_\perp}/k_\perp = \sin \psi_{p_\perp}/p_\perp = \sin \psi_{q_\perp}/q_\perp$. The wave kinetic equation (79) is now symmetric enough to apply a conformal transformation, called the Kuznetsov-Zakharov transformation. This transformation, a two-dimensional generalization of the Zakharov transformation, has been applied in various anisotropic problems (Kuznetsov, 1972; Balk et al. , 1990; Kuznetsov, 2001; Galtier and Bhattacharjee, 2003; Galtier, 2003). The bihomogeneity of the collision integrals in the wavenumbers k_\perp and k_\parallel allows us to use the transformation

$$\begin{aligned} p_\perp &\rightarrow k_\perp^2/p_\perp, \\ q_\perp &\rightarrow k_\perp q_\perp/p_\perp, \\ p_\parallel &\rightarrow k_\parallel^2/p_\parallel, \\ q_\parallel &\rightarrow k_\parallel q_\parallel/p_\parallel. \end{aligned} \quad (80)$$

We search for stationary solutions in the power law form $E(k_\perp, k_\parallel) \sim k_\perp^{-n} k_\parallel^{-m}$. (We will only consider positive parallel wavenumber.) The new form of the collision integral of equation (79), resulting from the summation of the integrand in its primary form and after the Kuznetsov-Zakharov transformation, is

$$\partial_t E^V(k_\perp, k_\parallel) = \quad (81)$$

$$\begin{aligned} & -\frac{\epsilon^2}{32} \int \sum_{s, s_p, s_q} \sin \psi_{q_\perp} (s k_\perp + s_p p_\perp + s_q q_\perp)^2 \left(\frac{s_p p_\perp - s_q q_\perp}{k_\parallel p_\perp q_\perp} \right)^2 s k_\parallel s_p p_\parallel \delta_{k_\parallel, p_\parallel q_\parallel} \delta(\Omega_{k,pq}) \\ & k_\perp^{-n} k_\parallel^{-m} p_\perp q_\perp^{-n} q_\parallel^{-m} \left(1 - \left(\frac{p_\perp}{k_\perp} \right)^{-n-1} \left(\frac{p_\parallel}{k_\parallel} \right)^{-m} \right) \left(1 - \left(\frac{p_\perp}{k_\perp} \right)^{2n-5} \left(\frac{p_\parallel}{k_\parallel} \right)^{2m-1} \right) \\ & dp_\perp dq_\perp dp_\parallel dq_\parallel. \end{aligned}$$

The above collision integral vanishes for specific values of n and m . The exact power law solutions correspond to these values. There are two different kind of solutions. The fluxless solution, also called the thermodynamic equilibrium solution, correspond to the equipartition state for which the flux of energy is zero. For this case, we have

$$\begin{aligned} n &= -1, \\ m &= 0. \end{aligned} \tag{82}$$

This result can easily be checked by direct substitution in the original wave kinetic equation. The most interesting solution of the wave kinetic equation (81) is the one for which the flux is non-zero and finite. The exact solution is called the Kuznetsov-Zakharov-Kolmogorov (KZK) spectrum and corresponds to the values,

$$\begin{aligned} n &= 5/2, \\ m &= 1/2. \end{aligned} \tag{83}$$

In other words, the KZK solution scales as

$$E^V(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-5/2} k_{\parallel}^{-1/2}. \tag{84}$$

We note that the power law scaling found is the same as the one for whistler wave turbulence. However here we are dealing with the kinetic energy not the magnetic one. A necessary condition for the realizability of the KZK spectra is that the turbulence be local in the sense that the behavior of the turbulence is determined primarily (but not only) by interaction between wave packets of comparable spatial scale (see *e.g.* [4]). To check *a posteriori* the validity of the solutions, we need to determine the domain of locality, *i.e.*, the domain where the collision integral converges. In practice, we check that the contribution of nonlocal interactions do not lead to a divergence of the collision integral. Here, the condition of locality is automatically satisfied since the anisotropic limit introduces in the problem a cut-off which prevents any infra-red divergence of the collision integral. This characteristic is also observed for internal gravity waves (Caillol and Zeitlin, 2000), inertial waves (Galtier, 2003) and whistler waves (Galtier and Bhattacharjee, 2003). The other consequence of the existence of such a cut-off is that it is not possible to evaluate precisely the Kolmogorov constant, *i.e.* the prefactor of the spectra. However, the sign of the energy transfer can be computed for a reasonable

range of cut-offs. We observe a positive sign which means that we have a direct energy cascade. Note that this information cannot be obtained through a heuristic reasoning but only with a rigorous analysis.

5.4 Master equations in the limit of pure Alfvén wave turbulence

In the large scale limit ($d_i k \rightarrow 0$) for which terms like $\xi_\Lambda^{-s^2}$ tend to 1, we note that an equipartition between the kinetic and magnetic energies may be obtained since their wave kinetic equations tend to be identical: if initially there is equipartition ($E^V(\mathbf{k}) = E^B(\mathbf{k}) = E(\mathbf{k})/2$) then at any time the equipartition will be conserved. The large scale limit corresponds to the standard MHD approximation for which the equipartition is in fact automatically satisfied by the kinematics. The large scale limit of expression (75), for a state of equipartition, gives

$$\partial_t E(\mathbf{k}) = \quad (85)$$

$$\begin{aligned} & \frac{\pi \epsilon^2}{256} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (s \Lambda k + s_p \Lambda_p p + s_q \Lambda_q q)^2 \\ & \left(\frac{s_q - s_p}{k_\parallel} \right)^2 s k_\parallel s_p p_\parallel E(\mathbf{q}) [E(\mathbf{p}) - E(\mathbf{k})] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}, \end{aligned}$$

with $\Omega_{k,pq} = B_0(s k_\parallel - s_p p_\parallel - s_q q_\parallel)$. As expected, only nonlinear interactions between Alfvén waves with different directional polarities $s_p = -s_q$ will contribute to the dynamics. After such a consideration and an expansion over the polarities, one finds

$$\partial_t E(\mathbf{k}) = \quad (86)$$

$$\frac{\pi \epsilon^2}{4 B_0} \int \sin^2 \psi_k (1 + \cos^2 \psi_q) p^2 E(\mathbf{q}) [E(\mathbf{p}) - E(\mathbf{k})] \delta(q_\parallel) \delta_{k,pq} d\mathbf{p} d\mathbf{q},$$

where we have used the identities

$$\begin{aligned} & \sum_{\Lambda, \Lambda_p, \Lambda_q} (\Lambda k + \Lambda_p p + \Lambda_q q)^2 (\Lambda k + \Lambda_p p - \Lambda_q q)^2 = \quad (87) \\ & 4 \left[((k+p)^2 - q^2)^2 + ((k-p)^2 - q^2)^2 \right] = 32 k^2 p^2 (1 + \cos^2 \psi_q), \end{aligned}$$

and the well known triangle relations. The angle ψ_q refers to the angle opposite to \mathbf{q} in the triangle defined by $\mathbf{k} = \mathbf{p} + \mathbf{q}$. (The same result can actually be found directly from equation (72); the inversion of the system made in section 5.1 is thus compatible with such a limit.) As noted before, the large scale limit of Hall MHD leads to the particular case where helicities are absent and where shear- and pseudo-Alfvén wave energies are the same. Equation (86) can be recovered from the general kinetic equations obtained by Galtier et al. (2000) when the same assumptions are made. The proof is given in Appendix F. Therefore, all the properties previously derived are recovered (scaling laws, direction of the cascade, Kolmogorov constant...). In particular, the energy spectrum follows the power law $E(k_\perp, k_\parallel) \sim k_\perp^{-2} f(k_\parallel)$, where f is an arbitrary function that is due to the dynamical decoupling of parallel planes in Fourier space. This is an exact solution of the wave kinetic equations.

6 Anisotropic heuristic description for Hall MHD

The study of wave turbulence in Hall MHD is a difficult task. We have seen above that for three specific limits we are able to find the exact power law energy spectra. Two of them (Alfvén and whistler wave turbulence) were already known and a heuristic description was given by Galtier et al. (2000) and Galtier and Bhattacharjee (2003). We shall derive here a generalized heuristic description able to recover the essential physics underlying the KZK spectra including the ion cyclotron wave turbulence. However, note that several phenomenologies may be used to take into account the various regimes involving a non trivial balance between propagation and nonlinear effects like for standard MHD (see *e.g.* Zhou et al., 2004).

We introduce anisotropy in our description by considering that

$$k \approx k_\perp \gg k_\parallel. \quad (88)$$

The primary variables used in the formalism are the generalized Elsässer variables \mathcal{Z}_Λ^s . Thus the nonlinear time built on the generalized Elsässer variables is

$$\tau_{NL} \sim \frac{1}{k_\perp \mathcal{Z}_\Lambda^s}. \quad (89)$$

Note here that \mathcal{Z}_Λ^s has a dimension of a velocity. In other words, it is *not* taken in Fourier space as it was introduced in Section 2.5. In a similar way, we find the following Hall MHD wave period

$$\tau_w \sim \frac{1}{\omega_\Lambda^s} = \frac{1}{-B_0 k_\parallel \xi_\Lambda^s}. \quad (90)$$

We introduce now the mean rate of energy dissipation per unit mass Π . Contrary to the electron MHD case, we do not have to renormalize this quantity since it is automatically taken into account by the generalized Elsässer variables. Then we have

$$\Pi \sim \frac{E}{\tau_{tr}} \sim \frac{E(k_\perp, k_\parallel) k_\perp k_\parallel}{\tau_{tr}}, \quad (91)$$

where the transfer time τ_{tr} has the usual form given by the wave kinetic equation

$$\tau_{tr} \sim \tau_{NL} \frac{\tau_{NL}}{\tau_w}; \quad (92)$$

it gives

$$\Pi \sim \frac{E(k_\perp, k_\parallel) k_\perp^3 \mathcal{Z}_\Lambda^{s^2}}{-B_0 \xi_\Lambda^s}. \quad (93)$$

To proceed further, we note that

$$(\mathcal{Z}_\Lambda^s)^2 \sim (\xi_\Lambda^s - \xi_\Lambda^{-s})^2 a_\Lambda^{s^2}, \quad (94)$$

and

$$E \sim (1 + \xi_\Lambda^{-s^2}) a_\Lambda^{s^2}. \quad (95)$$

In relations (94) and (95), both \mathcal{Z}_Λ^s and a_Λ^s are written in the physical space, *not* in Fourier space. By using the relationships given in Appendix A, we finally obtain

$$E(k_\perp, k_\parallel) \sim \sqrt{\Pi B_0} k_\perp^{-2} k_\parallel^{-1/2} (1 + k_\perp^2 d_i^2)^{-1/4}. \quad (96)$$

The heuristic prediction proposed here is able to describe anisotropic turbulence for the three different limits discussed above. We recover, in the small scale limit ($k_\perp d_i \rightarrow \infty$), the expected scaling law for whistler as well as ion cyclotron wave turbulence and, in the large scale limit ($k_\perp d_i \rightarrow 0$), the Alfvén wave turbulence scaling law (since then the parallel wavenumber

is a mute variable). The prediction is given for the total energy, therefore for the previous small scale limits one needs to consider only the magnetic or the kinetic energy respectively. One may understand this point by looking at equation (31) where the relation between the generalized Elsässer variables, the velocity and the magnetic field is given. In the small scale limit, we have for whistler waves $\xi_\Lambda^s \rightarrow -s d_i k$ and for ion cyclotron waves $\xi_\Lambda^s \rightarrow (-s d_i k)^{-1}$ (whereas in the large scale limit $\xi_\Lambda^s \rightarrow -s$); thus we see that either the kinetic or the magnetic field will dominate. Note that we ignore anisotropy and assume $k_\perp \sim k_\parallel \sim k$, we arrive at the scaling

$$E(k) \sim \sqrt{\Pi B_0} k^{-3/2} (1 + k^2 d_i^2)^{-1/4} \quad (97)$$

for the one-dimensional isotropic spectrum from which one recognizes the Iroshnikov (1963) and Kraichnan (1965) prediction for MHD.

To summarize our finding we propose the picture given in Fig. 2. It is a sketch of the power law energy spectrum predictions for perpendicular wavenumbers at a given k_\parallel . The energy spectrum of Hall MHD is characterized by two inertial ranges – the exact power law solutions of the wave kinetic equations – separated by a knee. The position of the knee corresponds to the scale where the Hall term becomes sub/dominant, *i.e.* when $k_\perp d_i \sim 1$. The heuristic prediction tries to make the link continuously (dashed line in Fig. 2) between these power laws but the heuristic spectrum may not be correct at intermediate scales since, in particular, Hall MHD turbulence is not necessarily anisotropic. However, we will show, in Section 8, that even at intermediate scales a moderate anisotropy is expected. The presence of a knee in the solar wind spectrum is well attested by *in situ* measurements of magnetic fluctuations (Coroniti et al., 1982; Denskat et al., 1983, Leamon et al., 1998; Bale et al., 2005). We will discuss about this point in Section 10; we will also comment the fact the comparison with observations is not direct, in particular, for the high frequency part of the spectrum. One reason is that the Taylor hypothesis, usually used at low frequency, is not applicable anymore.

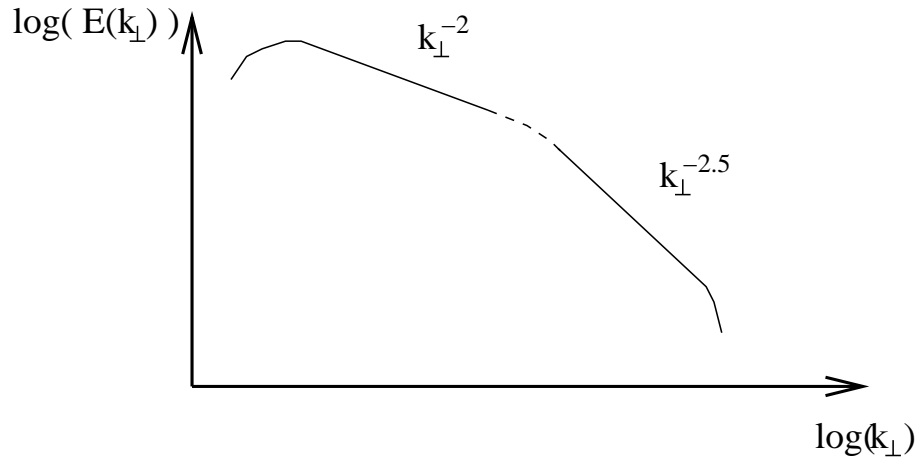


Figure 2: Sketch of the power law spectrum, at a given k_{\parallel} , expected for incompressible Hall MHD in the regime of wave turbulence.

7 Nonlocal interactions

7.1 Introduction

Nonlocal interactions may play a significant role in turbulence (see *e.g.* Laval et al., 2001). By nonlocal interactions, we mean interactions between well-separated scales. For triadic processes, it corresponds to the interaction of two highly elongated wavevectors with a third small wavevector. In Hall MHD the study of nonlocal interactions and nonlocal transfers are particularly interesting since the turbulent flow behaves very differently at different scales (see Mininni et al., 2005). At the larger scales it follows the MHD dynamics whereas at the smallest scales the flow may follow, for example, the EMHD dynamics. For simplicity, we will focus our analysis on the master equations (75). Therefore we will neglect the helicity effects and keep only the kinetic and magnetic energy terms. According to the situation, we will consider two different kinds of nonlocal interactions, namely $k \ll p, q$ or $p \ll k, q$ (see Fig. 3). In practice, small scale and large scale effects will be effective when, respectively, the limits $kd_i \rightarrow +\infty$ and $kd_i \rightarrow 0$ will be taken. Therefore, we see that large scale effects (small wavenumbers) will be driven by Alfvén waves (A) whereas small scale effects (large wavenumbers) will be driven by whistler (W) or ion cyclotron (C) waves. Note that we will not study nonlocal interactions between the same type of waves (AAA,

WWW or CCC interactions) which are somewhat more restrictive. Under such a condition, three different families of interactions may happen, namely WWA, CCA and WCA. We describe below such interactions.

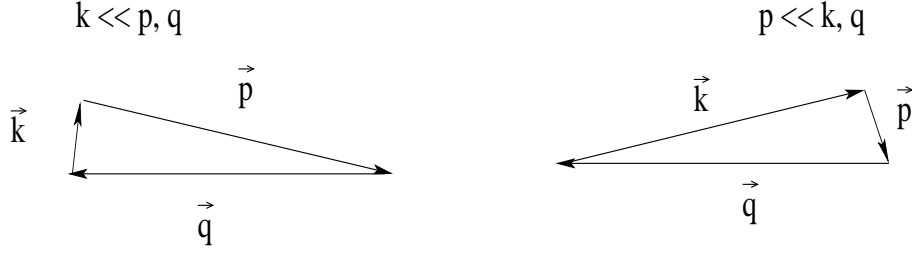


Figure 3: Nonlocal triadic interactions.

7.2 WWA interactions

First we shall consider the strongly nonlocal interactions of two whistler waves on one Alfvén wave. In other words, it means that the Alfvén wave will be supported by the wavevector \mathbf{k} , with $k \ll p, q$. For such a limit the master equations (75) reduce to

$$\partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = \quad (98)$$

$$\frac{\pi \epsilon^2}{8 d_i^2 B_0^2} \int \sum_{\Lambda, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \frac{(\Lambda k + s_p p + s_q q)^2}{2} \left(\frac{s_q q d_i - s_p p d_i}{k_{\parallel}} \right)^2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\frac{B_0^2 s k_{\parallel} s_p p_{\parallel} p d_i}{2} E^B(\mathbf{q}) \left[E^B(\mathbf{p}) - \left(\frac{E^V(\mathbf{k}) - E^B(\mathbf{k})}{-s \Lambda d_i k} \right) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

Note that for whistler waves the polarizations are fixed, *i.e.* $\Lambda_p = s_p$ and $\Lambda_q = s_q$. Further simplifications are possible and one obtains finally

$$\partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = \quad (99)$$

$$\frac{\pi \epsilon^2}{8} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q q - s_p p}{k_{\parallel}} \right)^2 (s_p p + s_q q) k_{\parallel} s_p p_{\parallel} p \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$E^B(\mathbf{q}) \left[E^V(\mathbf{k}) - E^B(\mathbf{k}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

A very interesting result is obtained: at leading order, the Alfvén wave is only affected by small scale whistler waves when the local (*i.e.* at a given wavevector \mathbf{k}) equipartition between the alfvénic kinetic and magnetic energies is broken, *i.e.* when $E^V(\mathbf{k}) \neq E^B(\mathbf{k})$. Additionally, we see that these nonlocal effects are the same for the alfvénic kinetic and the magnetic energies. This is not so trivial since whistler waves are described only by the magnetic energy. We also note that the collisional integral is linearly dependent on the difference between the large scale kinetic and magnetic energies which can be extracted from the integral. These comments may be summarized by the relation

$$\partial_t \left\{ \frac{E^V(\mathbf{k})}{E^B(\mathbf{k})} \right\} = I_1(\mathbf{k}) \left[E^V(\mathbf{k}) - E^B(\mathbf{k}) \right] \left\{ \frac{1}{1} \right\}, \quad (100)$$

where $I_1(\mathbf{k})$ is a real function that measures the effects of whistler waves.

The second case that we shall consider is the one where a whistler wave is affected by the interaction between an Alfvén wave and another whistler wave. In other words, it means that the Alfvén wave will be supported by the wavevector \mathbf{p} with $p \ll k, q$. The same kind of manipulations as before lead finally to the following wave kinetic equations

$$\begin{aligned} \partial_t \left\{ \frac{E^V(\mathbf{k})}{E^B(\mathbf{k})} \right\} = & \quad (101) \\ \frac{\pi \epsilon^2}{4} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{sk - s_q q}{p_{\parallel}} \right)^2 (sk + s_q q) sk_{\parallel} kp_{\parallel} \left\{ \frac{\frac{1}{d_i^2 k^2}}{1} \right\} \\ & E^B(\mathbf{q}) \left[E^B(\mathbf{p}) - E^V(\mathbf{p}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

As expected, we note that at leading order the kinetic energy of whistler waves is negligible and only the magnetic energy is relevant. We see that the Alfvén wave affects the whistler wave only when the local equipartition is broken. Thus for any kind of WWA nonlocal interactions, one needs a discrepancy from the alfvénic equipartition to have a non trivial dynamics.

7.3 CCA interactions

We study here the strongly nonlocal interactions of two ion cyclotron waves on one Alfvén wave: in other words, it means that the Alfvén wave will be

supported by the wavevector \mathbf{k} , with $k \ll p, q$. For such a limit the wave kinetic equations (75) write

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \quad (102)$$

$$\begin{aligned} & \frac{\pi \epsilon^2}{8 d_i^2 B_0^2} \int \sum_{\Lambda} \left(\frac{\sin \psi_k}{k} \right)^2 \frac{(\Lambda k - s_p p - s_q q)^2 d_i^8 p^4 q^4}{2 d_i^4 p^2 q^2} \left(\frac{s_q / q d_i - s_p / p d_i}{k_{\parallel}} \right)^2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \\ & \frac{B_0^2 s k_{\parallel} s_p p_{\parallel}}{2 p d_i} E^V(\mathbf{q}) \left[E^V(\mathbf{p}) - \left(\frac{E^V(\mathbf{k}) - E^B(\mathbf{k})}{-s \Lambda d_i k} \right) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

For ion cyclotron waves the polarizations are $\Lambda_p = -s_p$ and $\Lambda_q = -s_q$. Further simplifications lead to

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = \quad (103)$$

$$\begin{aligned} & \frac{\pi \epsilon^2}{8 d_i^2} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q / q - s_p / p}{k_{\parallel}} \right)^2 (s_p p + s_q q) p q^2 k_{\parallel} s_p p_{\parallel} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \\ & E^V(\mathbf{q}) \left[E^B(\mathbf{k}) - E^V(\mathbf{k}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

As before, we see that at leading order the Alfvén wave is only affected by small scale ion cyclotron waves when the local equipartition between the alfvénic kinetic and magnetic energies is broken. Additionally, we see that these nonlocal effects are the same for the alfvénic kinetic and magnetic energies. This is not so trivial since ion cyclotron waves are described only by the kinetic energy. As before, we note that the collisional integral is linearly dependent on the difference between the large scale kinetic and magnetic energies which can be extracted from the integral. These comments may be summarized by the relation

$$\partial_t \begin{Bmatrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{Bmatrix} = I_2(\mathbf{k}) \left[E^B(\mathbf{k}) - E^V(\mathbf{k}) \right] \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad (104)$$

where $I_2(\mathbf{k})$ is a real function that measures the effects of ion cyclotron waves.

The second case that we shall consider is the one where an ion cyclotron wave is affected by the interaction between an Alfvén wave and another ion

cyclotron wave. In other words, it means that the Alfvén wave will be supported by the wavevector \mathbf{p} with $p \ll k, q$. The same kind of manipulations as before lead finally to

$$\begin{aligned} \partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = & \quad (105) \\ \frac{\pi \epsilon^2}{4 d_i^4} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q/q - s/k}{p_{\parallel}} \right)^2 (sk + s_q q) \frac{sk_{\parallel} p_{\parallel} q^2}{k} \left\{ \begin{matrix} d_i^2 k^2 \\ 1 \end{matrix} \right\} \\ E^V(\mathbf{q}) \left[E^V(\mathbf{p}) - E^B(\mathbf{p}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

As expected, we note that at leading order the magnetic energy of ion cyclotron waves is negligible and only the kinetic energy is relevant. We see that the Alfvén wave affects the ion cyclotron wave only when its local equipartition is broken. Thus for any kind of CCA nonlocal interactions, one needs a discrepancy from the alfvénic equipartition to obtain a non trivial dynamics.

7.4 WCA interactions

First, we investigate the strongly nonlocal interactions of a whistler wave and an ion cyclotron wave on an Alfvén wave: in this case the Alfvén wave will be supported by the wavevector \mathbf{k} ; the whistler and ion cyclotron waves will be associated respectively to \mathbf{p} and \mathbf{q} , with $k \ll p, q$. For such a limit the wave kinetic equations (75) write

$$\begin{aligned} \partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = & \quad (106) \\ \frac{\pi \epsilon^2}{8 d_i^2 B_0^2} \int \sum_{\Lambda} \left(\frac{\sin \psi_k}{k} \right)^2 \frac{(\Lambda k + s_p p - s_q q)^2 (1 - q^2/p^2)^2}{2 d_i^2 q^2} \left(\frac{s_p p d_i - s_q/q d_i}{k_{\parallel}} \right)^2 \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \\ \frac{B_0^2 s k_{\parallel} s_p p_{\parallel} p d_i}{2} E^V(\mathbf{q}) \left[E^B(\mathbf{p}) - \left(\frac{E^V(\mathbf{k}) - E^B(\mathbf{k})}{-s \Lambda d_i k} \right) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

Further simplifications give finally

$$\begin{aligned} \partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = & \quad (107) \\ \frac{\pi \epsilon^2}{8 d_i^4} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_p p d_i - s_q/q d_i}{k_{\parallel}} \right)^2 (1 - q^2/p^2)^2 (s_p p - s_q q) \frac{k_{\parallel} s_p p_{\parallel} p}{q^2} \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\} \end{aligned}$$

$$E^V(\mathbf{q}) \left[E^V(\mathbf{k}) - E^B(\mathbf{k}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

As before, at leading order the Alfvén wave is only affected by small scales effects when the local alfvénic equipartition is broken. These small scale effects are only due to ion cyclotron waves. The reason of this apparent asymmetry is actually due to the initial association between wavevectors and waves (see below). We see that these nonlocal effects are the same for the alfvénic kinetic and the magnetic energies. Once again, we note that the collisional integral is linearly dependent on the difference between the large scale kinetic and magnetic energies which can be extracted from the integral. Then one can write

$$\partial_t \left\{ \frac{E^V(\mathbf{k})}{E^B(\mathbf{k})} \right\} = I_3(\mathbf{k}) \left[E^V(\mathbf{k}) - E^B(\mathbf{k}) \right] \left\{ \frac{1}{1} \right\}, \quad (108)$$

where $I_3(\mathbf{k})$ is a real function that measures the small scale effects due to the ion cyclotron wave.

Because of the apparent asymmetry mentioned above, we will also consider the situation where the whistler wave is associated to the wavevector \mathbf{q} and the ion cyclotron wave to the wavevector \mathbf{p} . For such a limit, we obtain in the same way as before the following wave kinetic equations

$$\partial_t \left\{ \frac{E^V(\mathbf{k})}{E^B(\mathbf{k})} \right\} = \quad (109)$$

$$\frac{\pi \epsilon^2}{8 d_i^6} \int \sum_{s,s_p,s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q q d_i - s_p / p d_i}{k_{\parallel}} \right)^2 (1 - p^2 / q^2)^2 (s_p p - s_q q) \frac{k_{\parallel} s_p p_{\parallel}}{p^3} \left\{ \frac{1}{1} \right\} \\ E^B(\mathbf{q}) \left[E^B(\mathbf{k}) - E^V(\mathbf{k}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

The same comments as before hold: at leading order the Alfvén wave is only affected by small scales effects when the local alfvénic equipartition is broken. Here these small scale effects are only due to whistler waves. We also see that these nonlocal effects are the same for the alfvénic kinetic and the magnetic energies. Once again, we note that the collisional integral is linearly dependent on the difference between the large scale kinetic and magnetic energies. Then we can write

$$\partial_t \left\{ \frac{E^V(\mathbf{k})}{E^B(\mathbf{k})} \right\} = I_4(\mathbf{k}) \left[E^B(\mathbf{k}) - E^V(\mathbf{k}) \right] \left\{ \frac{1}{1} \right\}, \quad (110)$$

where $I_4(\mathbf{k})$ is a real function that measures the small scale effects due to the whistler wave.

The next situation that we are going to investigate is the one where an ion cyclotron wave is affected by the interaction between an Alfvén wave and a whistler wave. In other words, it means that the Alfvén wave will be supported by the wavevector \mathbf{p} , the whistler and ion cyclotron waves will be associated respectively to \mathbf{q} and \mathbf{k} , with $p \ll k, q$. The same type of manipulations as before lead to

$$\partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = \quad (111)$$

$$\frac{\pi \epsilon^2}{4 d_i^8} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q q d_i - s/k d_i}{p_{\parallel}} \right)^2 \left(1 - k^2/q^2 \right)^2 (sk - s_q q) \frac{sk_{\parallel} p_{\parallel}}{k^5} \left\{ \begin{matrix} d_i^2 k^2 \\ 1 \end{matrix} \right\} \\ E^B(\mathbf{q}) \left[E^V(\mathbf{p}) - E^B(\mathbf{p}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

As expected, at leading order the magnetic energy of ion cyclotron waves is negligible and only the kinetic energy is relevant. Once again, the Alfvén wave affects the ion cyclotron wave only when the local equipartition is broken.

The last case is the one where a whistler wave is affected by the interaction between an Alfvén wave and an ion cyclotron wave. In other words, it means that the Alfvén wave will be supported by the wavevector \mathbf{p} , the whistler and ion cyclotron waves will be associated respectively to \mathbf{k} and \mathbf{q} , with $p \ll k, q$. Under these conditions and after some simplifications the master equations write

$$\partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = \quad (112)$$

$$\frac{\pi \epsilon^2}{4 d_i^4} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q/q d_i - sk d_i}{p_{\parallel}} \right)^2 \left(1 - q^2/k^2 \right)^2 (sk - s_q q) \frac{sk_{\parallel} k p_{\parallel}}{q^2} \left\{ \begin{matrix} 1/d_i^2 k^2 \\ 1 \end{matrix} \right\} \\ E^V(\mathbf{q}) \left[E^B(\mathbf{p}) - E^V(\mathbf{p}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

As expected, at leading order the kinetic energy of whistler waves is negligible and only the magnetic energy is relevant. Once again, the Alfvén wave affects the whistler wave only when the local equipartition is broken. In conclusion, for any type of WCA nonlocal interactions, one needs a discrepancy from the alfvénic equipartition to obtain a non trivial dynamics.

7.5 Discussion

We have seen that for any kind of nonlocal interactions (WWA, CCA or WCA) a non trivial dynamics is present as long as the large scale kinetic and magnetic energies are different. This result reveals from the above detail analysis is new and may explain some solar wind features. Indeed at first sight, the solar wind may be seen as a MHD flow where the kinetic and magnetic energies are balanced. However, fluctuations around this equipartition state exist. The present work may explain how these fluctuations are maintained by small scales processes. Small scales may also be affected by what happens at the largest scales of the system. In this case Alfvén waves may be seen as a potential source of energy for small scales. We report in Appendix G the result of the nonlocal analysis made on WCC interactions.

8 Source of anisotropy

8.1 Local interactions

In previous sections, we have seen three different limits for which anisotropic turbulence develops more or less strongly, *i.e.* the small scale limit, with right or left polarization, and the large scale limit of Hall MHD. For intermediate wavenumbers the situation is more difficult to analyze. Indeed, the wave kinetic equations are in general too complex to give a simple answer. In this section, we shall see what happens when strongly local interactions are only taken into account. It is a situation where wavevectors have approximately the same amplitude, *i.e.* $k \approx p \approx q$ (we consider triangles, $\mathbf{k} = \mathbf{p} + \mathbf{q}$, approximately equilaterals). We are particularly interested by what happens at intermediate scales.

Our analyze may start from the master equations (75). The hypothesis of strongly local interactions does not lead to drastic simplifications; it seems therefore more interesting to look at the resonance condition (54) that appears in equations (75). The first case that we may investigate is the one for which waves have the same polarization (left or right). After simplifications, one finds the approximate relation

$$\left(\frac{s_p - s_q}{k_{\parallel}}\right)^2 \approx \left(\frac{s_q - s}{p_{\parallel}}\right)^2 \approx \left(\frac{s - s_p}{q_{\parallel}}\right)^2. \quad (113)$$

Necessarily two directional wave polarities will be equal which implies that one of the parallel wavenumbers will be (approximately) equal to zero. In other words, local interactions between the same kind of waves lead to anisotropic turbulence where small scales are preferentially generated perpendicular to the external magnetic field.

The second case that we shall analyze is the one where we have local interactions between waves with different polarizations (left and right). Then the resonance condition simplifies to

$$\begin{aligned} \left(\frac{s_p(s_p\Lambda_p + \sqrt{1 + 4/d_i^2 p^2}) - s_q(s_q\Lambda_q + \sqrt{1 + 4/d_i^2 q^2})}{k_{\parallel}} \right)^2 &\approx \\ \left(\frac{s_q(s_q\Lambda_q + \sqrt{1 + 4/d_i^2 q^2}) - s(s\Lambda + \sqrt{1 + 4/d_i^2 k^2})}{p_{\parallel}} \right)^2 &\approx \\ \left(\frac{s(s\Lambda + \sqrt{1 + 4/d_i^2 k^2}) - s_p(s_p\Lambda_p + \sqrt{1 + 4/d_i^2 p^2})}{q_{\parallel}} \right)^2. \end{aligned} \quad (114)$$

Several combinations are possible. We note that for the particular case where two waves have both the same polarization (left or right) and the same directional polarity, one parallel wavenumber will be (approximately) equal to zero. Thus, such a situation leads to the same type of anisotropic turbulence as before. A similar analysis applied to all the other cases does not lead to such a conclusion: an isotropic turbulence may be generated by the nonlinear dynamics.

The wave kinetic equations (75) are composed of a sum over the different polarities (Λ - and s -type). In other words, all the cases that we have mentioned above have to be taken into account to understand what happens for strongly local interactions. Both the mechanisms leading to anisotropy and isotropy are in competition and therefore it is likely that globally a moderate anisotropy (less strong than what happens in the limits of large scale or small scale Hall MHD) is generated at intermediate wavenumbers.

8.2 Discussion

The local interaction analysis gives some hints to understand globally the nonlinear dynamics of wave turbulence in Hall MHD. It is also instructive

to analyze the opposite limit, namely the limit of strongly nonlocal interactions. In Section 7, we have introduced the concept of nonlocal interactions and discussed about some properties. Information about anisotropy can be extracted from the different cases analyzed. More precisely, one needs to study the solution of the condition $\Omega_{k,pq} = 0$ that has to be satisfied at the resonance. The general tendency that we observe (we will not give a detail description of every cases) is that most of nonlocal interactions leads to anisotropic turbulence where small scales are preferentially generated perpendicular to the external magnetic field.

Both local and nonlocal interactions contribute to develop anisotropy more or less strongly. Additionally, we know that a direct energy cascade is predicted in the large scale limit (pure Alfvén wave turbulence) as well as in the small scale limit (whistler and ion cyclotron wave turbulence) of Hall MHD. Thus it is expected that a turbulent anisotropic spectrum generated, or developed, at large scale will spread out over wavenumbers, from $kd_i < 1$ to $kd_i > 1$, extending anisotropy to smaller scales. This scenario may be applicable to the solar wind for which various indirect lines of evidence, at low and high frequencies, show that waves propagate at large angles to the background magnetic field and that the power in fluctuations parallel to the background magnetic field is much less than the perpendicular one (Coroniti et al., 1982; Leamon et al., 1998).

9 Wave versus strong turbulence

9.1 Domain of validity of wave turbulence

Wave turbulence deals with asymptotic developments which are based on a time scale separation, with a transfer time assumed to be much larger than the wave period. A consequence of this assumption is that the theory is not valid uniformly in all of \mathbf{k} -space. In this section, we are going to evaluate the condition of validity of the present theory. Generally, one needs to start with the wave kinetic equations for which exact power law solutions exist. In our case, different solutions have been found for different limits but in general – for intermediate wavenumbers – no solution has been found. For that reason we are going to use a dimensional analysis and, in particular, the heuristic spectrum derived in Section 6. This spectrum is based on anisotropic arguments, therefore the condition that we are going to

write is only valid for anisotropic turbulence and might not be applicable at intermediate wavenumbers.

We remind that the nonlinear time and the wave period are respectively

$$\tau_{NL} \sim \frac{1}{\epsilon k_{\perp} \mathcal{Z}_{\Lambda}^s} \quad (115)$$

and

$$\tau_w \sim \frac{1}{\omega_{\Lambda}^s} = \frac{1}{-B_0 k_{\parallel} \xi_{\Lambda}^s}. \quad (116)$$

Note that we have included explicitly the small parameter ϵ in the nonlinear time. The transfer time τ_{tr} is built on these time scales and writes

$$\tau_{tr} \sim \tau_{NL} \frac{\tau_{NL}}{\tau_w}. \quad (117)$$

Then the asymptotic condition $\tau_{tr} \gg \tau_w$ is

$$\frac{-B_0 k_{\parallel} \xi_{\Lambda}^s}{\epsilon^2 k_{\perp}^2 \mathcal{Z}_{\Lambda}^s} \gg \frac{1}{-B_0 k_{\parallel} \xi_{\Lambda}^s}, \quad (118)$$

where \mathcal{Z}_{Λ}^s has a dimension of a velocity (*i.e.* it is not taken in Fourier space as it was introduced in Section 2.5). To proceed further we use relations (94)–(95) and we take into account the heuristic spectrum (96); one obtains (with $B_0 = 1$)

$$(\mathcal{Z}_{\Lambda}^s)^2 \sim \frac{(\xi_{\Lambda}^s - \xi_{\Lambda}^{-s})^2}{1 + \xi_{\Lambda}^{-s2}} k_{\perp}^{-1} k_{\parallel}^{1/2} (1 + k_{\perp}^2 d_i^2)^{-1/4}. \quad (119)$$

Then we insert relation (119) into equation (118). By using relation (126), we finally obtain the inequality

$$k_{\parallel} \gg \epsilon^{4/3} k_{\perp}^{2/3} (1 + k_{\perp}^2 d_i^2)^{1/2} (1 + \xi_{\Lambda}^{s2})^{-2/3}. \quad (120)$$

Equation (120) is the condition of validity of wave turbulence for Hall MHD at the level of three-wave interactions. This condition is *not* in contradiction with the anisotropic assumption ($k_{\perp} \gg k_{\parallel}$) since the small parameter ϵ is present. Thus we can always find a domain of the \mathbf{k} -space where the theory is valid. We note, in particular, that for whistler and ion cyclotron wave turbulence the condition of validity reduces, respectively, to $k_{\parallel} \gg \epsilon^{4/3} k_{\perp}^{1/3}$ and $k_{\parallel} \gg \epsilon^{4/3} k_{\perp}^{5/3}$. We remind that it is an evaluation based on anisotropic assumption and therefore it might not be valid at intermediate wavenumbers. In the prohibited region (small k_{\parallel}) other higher-order processes such as four-wave interaction have to be taken into account. Basically it is the domain where strong turbulence occurs.

9.2 The two-dimensional state

The wave turbulence theory developed in this paper describes the Hall MHD dynamics of three-dimensional incompressible waves. Indeed, the condition of validity found previously shows that the two-dimensional state, *i.e.* modes with $k_{\parallel} = 0$, is not described by the wave kinetic equations. Actually, if we look at equations (56), we see that the nonlinear transfers decrease (linearly) with k_{\parallel} . For the forbidden value $k_{\parallel} = 0$ the transfer is exactly null. We see therefore that the two-dimensional – slow – modes decouple from the three-dimensional Hall MHD waves. Such decoupling is not new; it is found in a variety of problems like for internal gravity waves (see *e.g.* Phillips, 1969), rotating stratified flows (see *e.g.* Bartello, 1995) or pure rotating flows (see *e.g.* Smith and Waleffe, 1999).

As we have already noticed in Section 4.5 the MHD limit is somewhat singular for Hall MHD. A signature of this singular behavior is the appearance of principal value terms in the wave kinetic equations. The consequence is that there is an important difference between Hall MHD and MHD: in the latter case the two-dimensional modes drive the three-dimensional dynamics. Therefore, in this particular case, we need information from the forbidden region to describe the dynamics in the domain of applicability of wave turbulence. Note that even in this case the wave turbulence theory at the level of three-wave interactions has been shown to be applicable (Galtier et al., 2000; Nazarenko et al., 2001, Bhattacharjee and Ng, 2001). For Hall MHD in general, higher order processes like four-wave interactions may lead to a coupling between the two-dimensional state and three-dimensional modes.

9.3 Strong Hall MHD turbulence

We shall derive now the equivalent heuristic prediction as in Section 6 but for strong turbulence. We would like to compare, at least at the level of the phenomenology, the wave and strong turbulence regimes.

We will assume that a moderate external magnetic field B_0 is applied and that it still affects the Hall MHD dynamics. In other words we will distinguish the wavenumber k_{\perp} from k_{\parallel} , with $k_{\perp} \gg k_{\parallel}$, and we suppose that the nonlinear time is of the order of the wave period. Then one has

$$\Pi \sim \frac{E}{\tau_{tr}} \sim \frac{E(k_{\perp}, k_{\parallel}) k_{\perp} k_{\parallel}}{\tau_{NL}}. \quad (121)$$

The equivalent manipulations as in Section 6 lead to the prediction

$$E(k_{\perp}, k_{\parallel}) \sim \Pi^{2/3} k_{\perp}^{-5/3} k_{\parallel}^{-1} (1 + \delta_{\Lambda s} k_{\perp} d_i)^{-2/3}, \quad (122)$$

where $\delta_{\Lambda s}$ is introduced to distinguish the electron MHD case ($\delta_{ss} = 1$) from the ion MHD case ($\delta_{ss} = 0$). The heuristic prediction gives the energy spectrum scaling laws for the three different limits of standard, electron and ion MHD as well as for intermediate wavenumbers. Contrary to the wave turbulence regime the prediction shows that EMHD and IMHD follow different power laws. If the external magnetic field is small enough one may ignore anisotropy and write $k_{\perp} \sim k_{\parallel} \sim k$; it gives

$$E(k) \sim \Pi^{2/3} k^{-5/3} (1 + \delta_{\Lambda s} k d_i)^{-2/3}, \quad (123)$$

for the one-dimensional isotropic spectrum. It is interesting to note that the EMHD prediction is the one found and observed in direct numerical simulations by Biskamp et al. (1999). Note that the Kolmogorov scaling found at large scales is an unavoidable result since we have used in fact the same phenomenology as Kolmogorov. Note finally that the scaling law predictions made for strong and wave turbulence in the presence of anisotropy may be described in the context of the generalized critical balance proposed by Galtier et al. (2005). In that context the solutions found here would be two particular cases of a family of solutions. The result obtained here has to be contrasted with those obtained by Krishan and Mahajan (2004,2005) for strong isotropic turbulence. In these papers, the authors give several predictions separately for the magnetic and kinetic energy spectra. In particular, they conclude that the whistler-type prediction is very different from the observations and evoke a possible stronger damping for whistler waves. In the present paper, it is thought that the damping (*e.g.* due to the resonance) affects mainly the ion cyclotron waves (see also Goldstein et al., 1994).

10 Conclusion

10.1 Summary

In this paper we have investigated the steepening of the magnetic fluctuation power law spectra observed in the solar wind for frequencies higher than 0.5 Hz. We have shown that the high frequency part of the spectrum may

be attributed to dispersive nonlinear processes present in the incompressible Hall MHD equations rather than pure dissipation. In that context, we have introduced the generalized Elsässer variables adapted to Hall MHD and developed a wave turbulence formalism based on a complex helical decomposition. Then, we have derived the wave kinetic equations for three dimensional incompressible Hall MHD turbulence at the level of three-wave interactions, in the presence of a strong external magnetic field, and in the most general case, *i.e.* with global and magnetic helicities. In such a regime, Hall MHD turbulence is mainly anisotropic with small scales preferentially generated perpendicular to the external magnetic field. In particular, strong anisotropy is developed at small scales (ion cyclotron/IMHD and whistler/EMHD wave turbulence) and large scales (pure Alfvén/MHD wave turbulence). For intermediate wavenumbers, *i.e.* $kd_i \approx 1$ only a moderate anisotropy may happen. We have found two different power law energy spectra that are exact solutions of the wave kinetic equations, with a steeper power law at smaller scales. To illustrate this finding, we have developed an anisotropic phenomenology that describes continuously the different scaling laws for the energy spectrum: the existence of a double inertial range is recovered with a knee at intermediate scales where the Hall term becomes (sub-)dominant. We have also shown that the large scale limit of Hall MHD, *i.e.* the standard MHD, is singular and leads to the appearance of principal value terms in the wave kinetic equations. We have analyzed the nonlocal interactions between Alfvén, whistler and ion cyclotron waves and we have shown that, at leading order, a non trivial dynamics is present only when a discrepancy from the equipartition between the large scale kinetic and magnetic energies of Alfvén waves happens. Finally, we have given the conditions of validity of wave turbulence in Hall MHD and compared the wave and strong turbulence scaling laws.

10.2 Discussion

Solar wind magnetic fluctuation power law spectra are commonly observed to have an index around $-5/3$ at frequencies lower than 0.5Hz and a steeper power law behavior at higher frequencies with a spectral index on average around -3 (Coroniti et al., 1982; Denskat et al., 1983; Leamon et al., 1998). The latest analysis made from Cluster spacecraft data has – however – reported a spectral index about -2.12 (Bale et al., 2005). These small scale fluctuations are also characterized by a bias of the polarization suggesting that they are likely to be right-hand polarized, outward propagating waves

(Goldstein et al., 1994). For these reasons, it is claimed that Alfvén – left circularly polarized – fluctuations may be suppressed by proton cyclotron damping and that the high frequency power law spectra are likely to consist of whistler waves (see *e.g.* Stawicki et al., 2001). Additionally, it is often observed that the high frequency fluctuations of the magnetic field are much smaller than the background magnetic field (see *e.g.* Stawicki et al., 2001). Therefore, a wave turbulence approach seems to be well adapted to this problem. It is well known that whistler waves are solution to the Hall MHD equations. We have seen in the regime of wave turbulence that when the Hall term is added to the MHD equations a nontrivial dynamics happens at small scales (*i.e.* when $kd_i > 1$) with, in particular, a steepening of the power law energy spectra. This behavior is similar to what is observed in the solar wind and may be seen as an indication that the steepening of the solar wind magnetic fluctuation power law spectra is mainly due to nonlinear processes rather than pure dissipation. Under this new interpretation, the resistive dissipation range of frequencies may be moved to frequencies higher than the electron cyclotron frequency.

The Taylor hypothesis, that allows to connect directly a frequency to a wavenumber, is widely used to interpret the single spacecraft solar wind data. It is thought that this approximation is well adapted to analyze, in particular, the low frequency part of the magnetic and velocity fluctuations. The subsequent interpretation is mainly relevant for isotropic media. However, there are evidences that anisotropy is present in the solar wind at high and low frequencies (see *e.g.* Belcher and Davis, 1971). From a theoretical point of view, it is well known that the presence of a strong magnetic field influences the MHD turbulent flows (see *e.g.* Pouquet, 1978; Montgomery and Turner, 1981; Shebalin et al. 1983; Goldreich and Sridhar, 1995; Ng and Bhattacharjee, 1996-1997; Galtier et al., 2000-2005). The main effect is that MHD turbulence becomes mainly bidimensional with a nonlinear transfer essentially perpendicular to its direction. Direct numerical simulations of $2\frac{1}{2}$ D compressible Hall MHD for high and low beta β plasma (Ghosh and Goldstein, 1997) have also displayed such an anisotropic property when a strong magnetic field is present. As we have seen, the model that we propose here is also able to exhibit such an anisotropy. The predictions that we have made are for anisotropic turbulence, *i.e.* a situation where we distinguish the wavenumber k_\perp from k_\parallel . Therefore, a proper comparison with observational data will be possible only when a three dimensional energy spectrum will be accessible. It is important to note that average effects may alter significantly

the power law scaling. An illustration of such effects may be given from the recent heuristic MHD predictions made by [39] who have generalized the concept of critical balance. For a medium where a strong or moderate magnetic field $B_0 \hat{\mathbf{e}}_{\parallel}$ is present, they predict the anisotropic energy spectrum $E(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-\alpha} k_{\parallel}^{-\beta}$, with $3\alpha + 2\beta = 7$. This model is able to describe both the strong and wave turbulence regimes as well as the transition between them; it also satisfies the critical balance relationship $k_{\parallel} \propto k_{\perp}^{2/3}$. It is very interesting to note that if one averages the heuristic spectrum to obtain the one dimensional counterpart $E(k_{\perp})$, one obtains systematically a scaling law in $k_{\perp}^{-5/3}$ for any family of solutions (α, β) . Actually, this remark might explain the apparent contradiction between, on the one hand, the presence of anisotropy in the solar wind and, on the other hand, the Kolmogorov $(-5/3)$ scaling law for the low frequency energy spectrum. Of course, other explanations of the $5/3$ -spectrum are possible; an example is given in Oughton and Matthaeus (2005). The direct comparison between theoretical predictions and *in situ* measurements of three dimensional energy spectra is particularly crucial for the high frequency part of the power spectrum for which the usual Taylor hypothesis that allows to connect directly the frequency to the wavenumber is not applicable anymore. Efforts are currently made with Cluster spacecraft data from which it is possible to extract the three dimensional magnetic turbulent spectra of the magnetosheath thanks to multipoint measurements and a k-filtering technique (see *e.g.* Sahraoui et al., 2004). As explained in [3], Cluster spacecraft may exit from the terrestrial magnetosphere to make solar wind measurements. The application of the k-filtering technique to the high frequency part of the solar wind magnetic fluctuations seems then possible. It may lead, for the first time, to a direct and rigorous comparison with a model prediction of solar wind. Note that using multi-spacecraft analysis, Matthaeus et al. (2005) have recently investigated some spatial correlations from two-point measurements.

The model proposed in this paper is able to recover some solar wind properties but several aspects have not been discussed. For example, it would be interesting to investigate the effects of asymmetry, *i.e.* the fact that outward propagating waves dominate in the solar wind. In the framework of incompressible MHD, we know that asymmetry changes the index of the power law spectra. Similarly, the indices found here may be affected by asymmetry; this question will be tackled in the future. In this paper, we have found that a turbulent state made of ion cyclotron waves may exist around

a fraction of the ion cyclotron frequency ω_{ci} , namely for a frequency around $\omega_R = (k_{\parallel}/k) \omega_{ci}$. The presence of a resonance at a frequency ω_R lower than ω_{ci} is rarely discussed in the literature (see White et al., 2002) since the analysis is generally focused on parallel propagations ($k_{\parallel} = k$) for which the resonance frequency is exactly the ion cyclotron frequency. As we have seen, in the presence of a strong external magnetic field, Hall MHD turbulence becomes mainly bidimensional with an energy spectrum mainly spreaded out over the wavenumbers k_{\perp} . In this case, the resonance frequency may appear at a small fraction of ω_{ci} (this fraction being even smaller for heavier mass ions). These remarks may be crucial to understand the solar coronal heating problem in which the coronal temperature is far beyond what one can predict by the resistive MHD approximation: although the Hall MHD model is a fluid model that does not describe the resonance between waves and particles and therefore the particle heating, it offers the possibility to evaluate the rate of particle heating by assuming that the turbulent energy of ion cyclotron waves are mainly transfer into heating. This point is currently investigated and will be reported elsewhere. In the context of the solar wind, the measure of the position of the knee in the magnetic fluctuation power law spectrum may be seen as a proxy to measure the solar wind anisotropy.

The Hall effect is relevant in many astrophysical problems to understand, for example, the presence of instabilities in protostellar disks (Balbus and Terquem, 2001), the magnetic field evolution in neutron star crusts (Goldreich and Reisenegger, 1992; Cumming et al, 2004) or impulsive magnetic reconnection (see *e.g.* Bhattacharjee, 2004). Small scale turbulence is also a key issue in a number of problems from the interstellar medium (see the review made by [32], and, [95]), to solar physics (see *e.g.* , [85]), magnetospheric physics (see *e.g.* the recent paper by [46]) and laboratory devices such as tokamaks (see *e.g.* Wild et al., 1981; Taylor, 1993). Therefore, it is likely that the present model will interest several other (astrophysical) problems.

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A Some useful relationships

It is convenient to note the following identities:

$$1 - \xi_\Lambda^{s^2} = k \Lambda d_i \xi_\Lambda^s, \quad (124)$$

$$\xi_\Lambda^s \xi_\Lambda^{-s} = -1, \quad (125)$$

$$\xi_\Lambda^s - \xi_\Lambda^{-s} = -s \sqrt{4 + k^2 d_i^2}, \quad (126)$$

$$\xi_\Lambda^s + \xi_\Lambda^{-s} = -k d_i \Lambda, \quad (127)$$

$$\Lambda \left(\frac{1 - \xi_\Lambda^{-s^4}}{k} \right) \left(\frac{\omega_\Lambda^s}{1 + \xi_\Lambda^{-s^2}} \right) = d_i k_\parallel. \quad (128)$$

B Derivation of the wave kinetic equations

The starting point of the derivation of the wave kinetic equations for incompressible Hall MHD is the fundamental equation (47). We write successively equations for the second and third-order moments,

$$\partial_t \langle a_\Lambda^s a_{\Lambda'}^{s'} \rangle = \quad (129)$$

$$\begin{aligned} & \frac{\epsilon}{4 d_i} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_\Lambda^{s^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} M_{\substack{\Lambda \Lambda_p \Lambda_q \\ s s_p s_q}} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda'}^{s'} \rangle e^{-i \Omega_{pq, k} t} \delta_{pq, k} d\mathbf{p} d\mathbf{q} \\ & + \\ & \frac{\epsilon}{4 d_i} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_{\Lambda'}^{s'^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} M_{\substack{\Lambda' \Lambda_p \Lambda_q \\ s' s_p s_q}} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda}^s \rangle e^{-i \Omega_{pq, k'} t} \delta_{pq, k'} d\mathbf{p} d\mathbf{q}, \end{aligned}$$

and

$$\partial_t \langle a_\Lambda^s a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle = \quad (130)$$

$$\begin{aligned} & \frac{\epsilon}{4 d_i} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_\Lambda^{s^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_\Lambda^s - \xi_\Lambda^{-s}} M_{\substack{\Lambda \Lambda_p \Lambda_q \\ s s_p s_q}} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle e^{-i \Omega_{pq, k} t} \delta_{pq, k} d\mathbf{p} d\mathbf{q} \\ & + \\ & \frac{\epsilon}{4 d_i} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_{\Lambda'}^{s'^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} M_{\substack{\Lambda' \Lambda_p \Lambda_q \\ s' s_p s_q}} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda}^s a_{\Lambda''}^{s''} \rangle e^{-i \Omega_{pq, k'} t} \delta_{pq, k'} d\mathbf{p} d\mathbf{q} \end{aligned}$$

$$+ \frac{\epsilon}{4 d_i} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_{\Lambda''}^{s'' 2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} M_{-k'' p q}^{\Lambda'' \Lambda_p \Lambda_q s'' s_p s_q} \langle a_{\Lambda_p}^{s_p} a_{\Lambda_q}^{s_q} a_{\Lambda}^s a_{\Lambda'}^{s'} \rangle e^{-i \Omega_{pq, k''} t} \delta_{pq, k''} d\mathbf{p} d\mathbf{q}.$$

We shall write asymptotic closure (Newell et al., 2001) for our system. For that, we basically need to write the fourth-order moment in terms of a sum of the fourth-order cumulant plus products of second order ones. The asymptotic closure depends on two ingredients: the first is the degree to which the linear waves interact to randomize phases; the second relies on the fact that the nonlinear regeneration of the third-order moment by the fourth-order moment in equation (130) depends more on the product of the second order moments than it does on the fourth order cumulant. The fourth-order moment decomposes into the sum of three products of second-order moments, and a fourth-order cumulant. The latter does not contribute to secular behavior, and among the other products one is absent because of the homogeneity assumption. If we use the symmetric relations (49)–(52) and perform wavevector integrations, summations over polarities and time integration, then equation (130) becomes:

$$\begin{aligned} \langle a_{\Lambda}^s a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle &= \frac{\epsilon}{4 d_i} \Delta(\Omega_{kk'k''}) \delta_{kk'k''} \\ &\{ \xi_{\Lambda}^{s 2} \left[\frac{\xi_{\Lambda''}^{s''} - \xi_{\Lambda'}^{s'}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} \left(M_{k k' k''}^{\Lambda \Lambda' \Lambda'' s' s'' s} \right)^* + \frac{\xi_{\Lambda'}^{s'} - \xi_{\Lambda''}^{s''}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} \left(M_{k k' k''}^{\Lambda \Lambda'' \Lambda' s'' s' s} \right)^* \right] q_{\Lambda'}^{s'} q_{\Lambda''}^{s''} \\ &+ \xi_{\Lambda'}^{s' 2} \left[\frac{\xi_{\Lambda''}^{s''} - \xi_{\Lambda}^s}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} \left(M_{k' k k''}^{\Lambda' \Lambda \Lambda'' s' s'' s} \right)^* + \frac{\xi_{\Lambda}^s - \xi_{\Lambda''}^{s''}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} \left(M_{k' k k''}^{\Lambda' \Lambda'' \Lambda s'' s' s} \right)^* \right] q_{\Lambda}^s q_{\Lambda''}^{s''} \\ &+ \xi_{\Lambda''}^{s'' 2} \left[\frac{\xi_{\Lambda}^s - \xi_{\Lambda'}^{s'}}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} \left(M_{k'' k k'}^{\Lambda'' \Lambda' \Lambda s' s'' s} \right)^* + \frac{\xi_{\Lambda'}^{s'} - \xi_{\Lambda}^s}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda''}^{-s''}} \left(M_{k'' k k'}^{\Lambda'' \Lambda \Lambda' s'' s' s} \right)^* \right] q_{\Lambda'}^{s'} q_{\Lambda}^s \} , \end{aligned} \quad (131)$$

where

$$\Delta(\Omega_{kk'k''}) = \int_0^t e^{i \Omega_{kk'k''} t'} dt' = \frac{e^{i \Omega_{kk'k''} t} - 1}{i \Omega_{kk'k''}}. \quad (132)$$

The introduction of symmetric relations (49)–(52) into (131) allows us to simplify further the previous equation; one obtains:

$$\langle a_{\Lambda}^s a_{\Lambda'}^{s'} a_{\Lambda''}^{s''} \rangle = \frac{\epsilon}{2 d_i} \Delta(\Omega_{kk'k''}) \delta_{kk'k''} \left(M_{k k' k''}^{\Lambda \Lambda' \Lambda''} \right)^* \quad (133)$$

$$\left[\xi_{\Lambda}^{s^2} \frac{\xi_{\Lambda''}^{s'} - \xi_{\Lambda'}^{s'}}{\xi_{\Lambda}^s - \xi_{\Lambda'}^{-s}} q_{\Lambda'}^{s'} q_{\Lambda''}^{s''} + \xi_{\Lambda'}^{s^2} \frac{\xi_{\Lambda}^s - \xi_{\Lambda''}^{s''}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda''}^{-s'}} q_{\Lambda}^s q_{\Lambda''}^{s''} + \xi_{\Lambda''}^{s^2} \frac{\xi_{\Lambda'}^{s'} - \xi_{\Lambda}^s}{\xi_{\Lambda''}^{s''} - \xi_{\Lambda}^{-s}} q_{\Lambda}^s q_{\Lambda'}^{s'} \right].$$

We insert expression (133) into equation (129); it leads to:

$$\begin{aligned} \partial_t q_{\Lambda}^s(\mathbf{k}) = & \quad (134) \\ & \frac{\epsilon^2}{8 d_i^2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_{\Lambda}^{s^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} \left| M_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \right|^2 \Delta(\Omega_{pq,k}) e^{-i\Omega_{pq,k}t} \delta_{pq,k} \\ & \left[\xi_{\Lambda_p}^{s_p^2} \frac{\xi_{\Lambda}^s - \xi_{\Lambda_q}^{s_q}}{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda_q}^{-s_p}} q_{\Lambda}^s q_{\Lambda_q}^{s_q} + \xi_{\Lambda_q}^{s_q^2} \frac{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda}^s}{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda}^{-s_q}} q_{\Lambda}^s q_{\Lambda_p}^{s_p} + \xi_{\Lambda}^{s^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda}^s - \xi_{\Lambda}^{-s}} q_{\Lambda_p}^{s_p} q_{\Lambda_q}^{s_q} \right] d\mathbf{p} d\mathbf{q} \\ & + \\ & \frac{\epsilon^2}{8 d_i^2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_{\Lambda'}^{s'^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} \left| M_{-k' p q}^{\Lambda' \Lambda_p \Lambda_q} \right|^2 \Delta(\Omega_{pq,k'}) e^{-i\Omega_{pq,k'}t} \delta_{pq,k'} \\ & \left[\xi_{\Lambda_p}^{s_p^2} \frac{\xi_{\Lambda'}^{s'} - \xi_{\Lambda_q}^{s_q}}{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda_q}^{-s_p}} q_{\Lambda'}^{s'} q_{\Lambda_q}^{s_q} + \xi_{\Lambda_q}^{s_q^2} \frac{\xi_{\Lambda_p}^{s_p} - \xi_{\Lambda'}^{s'}}{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_q}^{-s_q}} q_{\Lambda'}^{s'} q_{\Lambda_p}^{s_p} + \xi_{\Lambda'}^{s'^2} \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{\xi_{\Lambda'}^{s'} - \xi_{\Lambda'}^{-s'}} q_{\Lambda_p}^{s_p} q_{\Lambda_q}^{s_q} \right] d\mathbf{p} d\mathbf{q}. \end{aligned}$$

The long-time behavior of the wave kinetic equation (134) is given by the Riemman-Lebesgue Lemma which tells us that, for $t \rightarrow +\infty$, we have

$$e^{-ixt} \Delta(x) = \Delta(-x) \rightarrow \pi \delta(x) - i\mathcal{P}(1/x), \quad (135)$$

where \mathcal{P} is the principal value of the integral. The two terms of equation (134) are complex conjugated therefore if in the second term we replace the dummy integration variables \mathbf{p} , \mathbf{q} , by $-\mathbf{p}$, $-\mathbf{q}$, we can simplify further equation (134) since, in particular, principal value terms compensate exactly. Finally, we obtain the wave kinetic equations for incompressible Hall MHD:

$$\partial_t q_{\Lambda}^s(\mathbf{k}) = \quad (136)$$

$$\frac{\pi \epsilon^2}{4 d_i^2} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ s_p, s_q}} \xi_\Lambda^s \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{1 - \xi_\Lambda^{-s^2}} \left| M_{-k \ p \ q}^{\Lambda \Lambda_p \Lambda_q \ s \ s_p \ s_q} \right|^2 \delta(\Omega_{k,pq}) \delta_{k,pq} \\ \left[\xi_{\Lambda_p}^{s_p} \frac{\xi_\Lambda^s - \xi_{\Lambda_q}^{s_q}}{1 - \xi_{\Lambda_p}^{-s_p^2}} q_\Lambda^s q_{\Lambda_q}^{s_q} + \xi_{\Lambda_q}^{s_q} \frac{\xi_{\Lambda_p}^{s_p} - \xi_\Lambda^s}{1 - \xi_{\Lambda_q}^{-s_q^2}} q_\Lambda^s q_{\Lambda_p}^{s_p} + \xi_\Lambda^s \frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{1 - \xi_\Lambda^{-s^2}} q_{\Lambda_p}^{s_p} q_{\Lambda_q}^{s_q} \right] d\mathbf{p} d\mathbf{q},$$

where

$$\left| M_{-k \ p \ q}^{\Lambda \Lambda_p \Lambda_q \ s \ s_p \ s_q} \right|^2 = \left(\frac{\sin \psi_k}{k} \right)^2 (\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_\Lambda^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2.$$

The last step that we have to follow to obtain the same expression as (56) is to include the resonance relations (54) into the previous equations.

C Pseudo-dispersive MHD waves and complex helical decomposition

This section is devoted to the derivation of the wave kinetic equations for pure alfvénic turbulence ($kd_i = 0$). These equations were already derived by Galtier et al. (2000) but here we use the complex helicity decomposition. We will see that some differences appear in the kinematics that renders the MHD description somewhat singular (principal value terms appear) by opposition to the dispersive Hall MHD description. We start from the standard incompressible and inviscid MHD equations,

$$\nabla \cdot \mathbf{V} = 0, \quad (137)$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla P_* + \mathbf{B} \cdot \nabla \mathbf{B}, \quad (138)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V}, \quad (139)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (140)$$

The same notation as before is used. We introduce the fluctuating fields $\mathbf{B}(\mathbf{x}) = B_0 \hat{\mathbf{e}}_\parallel + \epsilon \mathbf{b}(\mathbf{x})$, $\mathbf{V}(\mathbf{x}) = \epsilon \mathbf{v}(\mathbf{x})$ and we Fourier transform the MHD equations. One obtains:

$$\partial_t \mathbf{z}^s_{\mathbf{k}} - i s k_\parallel B_0 \mathbf{z}^s_{\mathbf{k}} = -\epsilon \{ \mathbf{z}^{-s} \cdot \nabla \mathbf{z}^s - \nabla P_* \}_{\mathbf{k}}, \quad (141)$$

$$\mathbf{k} \cdot \mathbf{z}_{\mathbf{k}}^s = 0, \quad (142)$$

where we use the standard Elsässer variables $\mathbf{z}^s = \mathbf{v} + s\mathbf{b}$. The linear solution ($\epsilon = 0$) corresponds to linearly polarized Alfvén waves for which $\omega_k = B_0 k_{\parallel}$. We introduce the complex helicity basis and define

$$\mathbf{z}_{\mathbf{k}}^s = \sum_{\Lambda} \mathcal{Z}_{\Lambda}^s(\mathbf{k}) e^{is\omega_k t} \mathbf{h}_{\mathbf{k}}^{\Lambda} = \sum_{\Lambda} \mathcal{Z}_{\Lambda}^s e^{is\omega_k t} \mathbf{h}_{\mathbf{k}}^{\Lambda}. \quad (143)$$

We see that this decomposition is not natural for linear polarized Alfvén waves since for a given direction of propagation (a given s) we have two contributions for each value of Λ . We nevertheless use this decomposition to show the compatibility with the Hall MHD turbulence description. After the substitution of (143) into (141)–(142), we obtain

$$\partial_t \mathcal{Z}_{\Lambda}^s = -\frac{i\epsilon}{2} \int \sum_{\Lambda_p, \Lambda_q} \mathcal{Z}_{\Lambda_p}^{-s} \mathcal{Z}_{\Lambda_q}^s (\mathbf{k} \cdot \mathbf{h}_{\mathbf{p}}^{\Lambda_p}) (\mathbf{h}_{\mathbf{q}}^{\Lambda_q} \cdot \mathbf{h}_{\mathbf{k}}^{-\Lambda}) e^{-is(\omega_k + \omega_p - \omega_q)t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}. \quad (144)$$

Note that we do not have a summation over the directional polarity s_p and s_q . The physical reason is that the nonlinear coupling in incompressible MHD involves only Alfvén waves propagating in opposite directions. Thus the information about the direction of propagation is already taken into account in equation (141). We use the local decomposition and find after some algebra

$$\partial_t \mathcal{Z}_{\Lambda}^s = \frac{\epsilon}{4} \int \sum_{\Lambda_p, \Lambda_q} (k\Lambda + q\Lambda_q - p\Lambda_p) N_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \mathcal{Z}_{\Lambda_p}^{-s} \mathcal{Z}_{\Lambda_q}^s e^{-2isB_0 p_{\parallel} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \quad (145)$$

where

$$N_{k p q}^{\Lambda \Lambda_p \Lambda_q} = e^{i(\Lambda\Phi_k + \Lambda_p\Phi_p + \Lambda_q\Phi_q)} \Lambda \Lambda_p \Lambda_q \frac{\sin \psi_k}{k} (\Lambda k + \Lambda_p p + \Lambda_q q). \quad (146)$$

We note that the matrix N possesses the following properties ($*$ denotes the complex conjugate),

$$\left(N_{k p q}^{\Lambda \Lambda_p \Lambda_q} \right)^* = N_{k p q}^{-\Lambda -\Lambda_p -\Lambda_q} = N_{-k -p -q}^{\Lambda \Lambda_p \Lambda_q}, \quad (147)$$

$$N_{k p q}^{\Lambda \Lambda_p \Lambda_q} = -N_{k q p}^{\Lambda \Lambda_q \Lambda_p}, \quad (148)$$

$$N_{k p q}^{\Lambda \Lambda_p \Lambda_q} = -N_{q p k}^{\Lambda_q \Lambda_p \Lambda}, \quad (149)$$

$$N_{k p q}^{\Lambda \Lambda_p \Lambda_q} = -N_{p k q}^{\Lambda_p \Lambda \Lambda_q}. \quad (150)$$

Equation (145) is the fundamental equation that describes the slow evolution of the Alfvén wave amplitudes due to the nonlinear terms of the incompressible MHD equations. Note that (i) we have already used the resonance condition to simplify the coefficient in the complex exponential function; (ii) a comparison with the large scale limit of equation (47) is possible if we sum over the directional polarities. We follow the same steps as in Appendix B: we write successively equations for the second and third-order moments,

$$\partial_t \langle \mathcal{Z}_\Lambda^s \mathcal{Z}_{\Lambda'}^{s'} \rangle = \quad (151)$$

$$\begin{aligned} & \frac{\epsilon}{4} \int \sum_{\Lambda_p, \Lambda_q} (k\Lambda + q\Lambda_q - p\Lambda_p) N_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \langle \mathcal{Z}_{\Lambda_p}^{-s} \mathcal{Z}_{\Lambda_q}^s \mathcal{Z}_{\Lambda'}^{s'} \rangle e^{-2isB_0 p_{\parallel} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \\ & + \\ & \frac{\epsilon}{4} \int \sum_{\Lambda_p, \Lambda_q} (k'\Lambda' + q\Lambda_q - p\Lambda_p) N_{-k' p q}^{\Lambda' \Lambda_p \Lambda_q} \langle \mathcal{Z}_{\Lambda_p}^{-s'} \mathcal{Z}_{\Lambda_q}^{s'} \mathcal{Z}_{\Lambda}^s \rangle e^{-2is'B_0 p_{\parallel} t} \delta_{pq,k'} d\mathbf{p} d\mathbf{q}, \end{aligned}$$

and

$$\partial_t \langle \mathcal{Z}_\Lambda^s \mathcal{Z}_{\Lambda'}^{s'} \mathcal{Z}_{\Lambda''}^{s''} \rangle = \quad (152)$$

$$\begin{aligned} & \frac{\epsilon}{4} \int \sum_{\Lambda_p, \Lambda_q} (k\Lambda + q\Lambda_q - p\Lambda_p) N_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \langle \mathcal{Z}_{\Lambda_p}^{-s} \mathcal{Z}_{\Lambda_q}^s \mathcal{Z}_{\Lambda'}^{s'} \mathcal{Z}_{\Lambda''}^{s''} \rangle e^{-2isB_0 p_{\parallel} t} \delta_{pq,k} d\mathbf{p} d\mathbf{q}, \\ & + \\ & \frac{\epsilon}{4} \int \sum_{\Lambda_p, \Lambda_q} (k'\Lambda' + q\Lambda_q - p\Lambda_p) N_{-k' p q}^{\Lambda' \Lambda_p \Lambda_q} \langle \mathcal{Z}_{\Lambda_p}^{-s'} \mathcal{Z}_{\Lambda_q}^{s'} \mathcal{Z}_{\Lambda}^s \mathcal{Z}_{\Lambda''}^{s''} \rangle e^{-2is'B_0 p_{\parallel} t} \delta_{pq,k'} d\mathbf{p} d\mathbf{q}, \\ & + \\ & \frac{\epsilon}{4} \int \sum_{\Lambda_p, \Lambda_q} (k''\Lambda'' + q\Lambda_q - p\Lambda_p) N_{-k'' p q}^{\Lambda'' \Lambda_p \Lambda_q} \langle \mathcal{Z}_{\Lambda_p}^{-s''} \mathcal{Z}_{\Lambda_q}^{s''} \mathcal{Z}_{\Lambda}^s \mathcal{Z}_{\Lambda'}^{s'} \rangle e^{-2is''B_0 p_{\parallel} t} \delta_{pq,k''} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

We define the density tensor $q_{\Lambda\Lambda'}^{ss'}(\mathbf{k})$ for an homogeneous turbulence,

$$\langle \mathcal{Z}_\Lambda^s(\mathbf{k}) \mathcal{Z}_{\Lambda'}^{s'}(\mathbf{k}') \rangle \equiv q_{\Lambda\Lambda'}^{ss'}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{k}') \delta_{ss'}. \quad (153)$$

The presence of the delta $\delta_{ss'}$ means that correlations with opposite polarities have no long-time influence in the wave turbulence regime; the second delta distribution $\delta(\mathbf{k} + \mathbf{k}')$ is the consequence of the homogeneity assumption. We note that the kinematics does not impose any condition about the polarization Λ . The reason is that Λ does not appear in the Alfvén wave frequency. This remark shows a fundamental difference with the Hall MHD case. As we will see it is the reason why principal value terms appear in MHD but not

in Hall MHD. After the same kind of manipulations as in Appendix B, the third-order moment equation (152) becomes:

$$\begin{aligned}
\langle \mathcal{Z}_\Lambda^s \mathcal{Z}_{\Lambda'}^{s'} \mathcal{Z}_{\Lambda''}^{s''} \rangle = & \quad (154) \\
\frac{\epsilon}{4} \delta_{kk'k''} \{ & \sum_{\Lambda_p, \Lambda_q} (k\Lambda + k''\Lambda_q - k'\Lambda_p) \left(N_{k'k''k}^{\Lambda\Lambda_p\Lambda_q} \right)^* \Delta(2sB_0k_{\parallel}) q_{\Lambda_p\Lambda'}^{-s-s} q_{\Lambda_q\Lambda''}^{ss} \\
& + \\
& \sum_{\Lambda_p, \Lambda_q} (k\Lambda + k'\Lambda_q - k''\Lambda_p) \left(N_{k'k''k}^{\Lambda\Lambda_p\Lambda_q} \right)^* \Delta(2sB_0k_{\parallel}) q_{\Lambda_p\Lambda''}^{-s-s} q_{\Lambda_q\Lambda'}^{ss} \\
& + \\
& \sum_{\Lambda_p, \Lambda_q} (k'\Lambda' + k''\Lambda_q - k\Lambda_p) \left(N_{k'k''k}^{\Lambda'\Lambda_p\Lambda_q} \right)^* \Delta(2s'B_0k_{\parallel}) q_{\Lambda_p\Lambda}^{-s'-s'} q_{\Lambda_q\Lambda''}^{s's'} \\
& + \\
& \sum_{\Lambda_p, \Lambda_q} (k'\Lambda' + k\Lambda_q - k''\Lambda_p) \left(N_{k'k''k}^{\Lambda'\Lambda_p\Lambda_q} \right)^* \Delta(2s'B_0k_{\parallel}) q_{\Lambda_p\Lambda''}^{-s'-s'} q_{\Lambda_q\Lambda}^{s's'} \\
& + \\
& \sum_{\Lambda_p, \Lambda_q} (k''\Lambda'' + k'\Lambda_q - k\Lambda_p) \left(N_{k''k'k}^{\Lambda''\Lambda_p\Lambda_q} \right)^* \Delta(2s''B_0k_{\parallel}) q_{\Lambda_p\Lambda}^{-s''-s''} q_{\Lambda_q\Lambda'}^{s''s''} \\
& + \\
& \sum_{\Lambda_p, \Lambda_q} (k''\Lambda'' + k\Lambda_q - k'\Lambda_p) \left(N_{k''k'k}^{\Lambda''\Lambda_p\Lambda_q} \right)^* \Delta(2s''B_0k_{\parallel}) q_{\Lambda_p\Lambda'}^{-s''-s''} q_{\Lambda_q\Lambda}^{s''s''} \}.
\end{aligned}$$

We insert expression (154) into (151). We note that only the third and the fifth terms of expression (154) will contribute. One obtains

$$\begin{aligned}
\partial_t q_{\Lambda\Lambda'}^{ss}(\mathbf{k}) = & \quad (155) \\
\frac{\epsilon}{16} \{ & \int \sum_{\substack{\Lambda_p, \Lambda_q \\ \bar{\Lambda}_p, \bar{\Lambda}_q}} (k\Lambda + q\Lambda_q - p\Lambda_p) (q\Lambda_q + k\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{-k p q}^{\Lambda\Lambda_p\Lambda_q} \left(-N_{-k p q}^{\bar{\Lambda}_q\bar{\Lambda}_p\Lambda_q} \right)^* \\
& \Delta(-2sB_0p_{\parallel}) q_{\Lambda_p\Lambda_p}^{-s-s} q_{\Lambda_q\Lambda'}^{ss} \delta_{pq,k} d\mathbf{p} d\mathbf{q} \\
& +
\end{aligned}$$

$$\begin{aligned}
& \int \sum_{\substack{\Lambda_p, \Lambda_q \\ \bar{\Lambda}_p, \bar{\Lambda}_q}} (k\Lambda + q\Lambda_q - p\Lambda_p) (k\Lambda' + q\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \left(N_{-k p q}^{\Lambda \bar{\Lambda}_p \bar{\Lambda}_q} \right)^* \\
& \quad \Delta(-2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \delta_{pq,k} d\mathbf{p} d\mathbf{q} \\
& \quad + \\
& \int \sum_{\substack{\Lambda_p, \Lambda_q \\ \bar{\Lambda}_p, \bar{\Lambda}_q}} (k\Lambda' + q\Lambda_q - p\Lambda_p) (q\Lambda_q + k\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{k p q}^{\Lambda' \Lambda_p \Lambda_q} \left(-N_{k p q}^{\bar{\Lambda}_q \bar{\Lambda}_p \Lambda_q} \right)^* \\
& \quad \Delta(-2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \delta_{kpq} d\mathbf{p} d\mathbf{q} \\
& \quad + \\
& \int \sum_{\substack{\Lambda_p, \Lambda_q \\ \bar{\Lambda}_p, \bar{\Lambda}_q}} (k\Lambda' + q\Lambda_q - p\Lambda_p) (k\Lambda + q\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{k p q}^{\Lambda' \Lambda_p \Lambda_q} \left(N_{k p q}^{\Lambda \bar{\Lambda}_p \bar{\Lambda}_q} \right)^* \\
& \quad \Delta(-2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \delta_{kpq} d\mathbf{p} d\mathbf{q} \}.
\end{aligned}$$

We change signs for wavevectors \mathbf{p} and \mathbf{q} in the last two terms and obtain:

$$\begin{aligned}
& \partial_t q_{\Lambda \Lambda'}^{ss}(\mathbf{k}) = \frac{\epsilon}{16} \int \sum_{\substack{\Lambda_p, \Lambda_q \\ \bar{\Lambda}_p, \bar{\Lambda}_q}} \quad (156) \\
& \{ (k\Lambda + q\Lambda_q - p\Lambda_p) (q\Lambda_q + k\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \left(-N_{-k p q}^{\bar{\Lambda}_q \bar{\Lambda}_p \Lambda_q} \right)^* \Delta(-2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \\
& \quad + \\
& (k\Lambda + q\Lambda_q - p\Lambda_p) (k\Lambda' + q\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{-k p q}^{\Lambda \Lambda_p \Lambda_q} \left(N_{-k p q}^{\Lambda \bar{\Lambda}_p \bar{\Lambda}_q} \right)^* \Delta(-2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \\
& \quad + \\
& (k\Lambda' + q\Lambda_q - p\Lambda_p) (q\Lambda_q + k\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{-k p q}^{\bar{\Lambda}_q \bar{\Lambda}_p \Lambda_q} \left(-N_{-k p q}^{\Lambda' \Lambda_p \Lambda_q} \right)^* \Delta(2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \\
& \quad + \\
& (k\Lambda' + q\Lambda_q - p\Lambda_p) (k\Lambda + q\bar{\Lambda}_q - p\bar{\Lambda}_p) N_{-k p q}^{\Lambda \bar{\Lambda}_p \bar{\Lambda}_q} \left(N_{-k p q}^{\Lambda' \Lambda_p \Lambda_q} \right)^* \Delta(2sB_0 p_{\parallel}) q_{\bar{\Lambda}_p \Lambda_p}^{-s-s} q_{\bar{\Lambda}_q \Lambda_q}^{ss} \} \\
& \quad \delta_{pq,k} d\mathbf{p} d\mathbf{q}.
\end{aligned}$$

The long-time behavior is then given by the Riemman-Lebesgue Lemma:

$$\Delta(\pm 2sB_0p_{\parallel}) \rightarrow \pi\delta(2sB_0p_{\parallel}) \mp i\mathcal{P}(1/2sB_0p_{\parallel}). \quad (157)$$

We see that principal value terms will appear in the long-time limit. The reason is that the polarization Λ and Λ' in the density tensor $q_{\Lambda\Lambda'}^{ss}(\mathbf{k})$ are not the same in general. Therefore we lose the symmetry between terms that we had in the Hall MHD case (see Appendix B).

D Simplified MHD wave kinetic equations

In in section, we continue the analysis made in Appendix C when only terms symmetric in Λ are retained, *i.e.* terms like $q_{\Lambda\Lambda}^{ss}$. Then expression (156) simplifies to

$$\begin{aligned} \partial_t q_{\Lambda\Lambda}^{ss}(\mathbf{k}) = & \frac{\epsilon}{16} \int \sum_{\Lambda_p, \Lambda_q} \quad (158) \\ & \{ (k\Lambda + q\Lambda_q - p\Lambda_p)^2 N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \left(-N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \right)^* \Delta(-2sB_0p_{\parallel}) q_{\Lambda_p\Lambda_p}^{-s-s} q_{\Lambda\Lambda}^{ss} \\ & + \\ & (k\Lambda + q\Lambda_q - p\Lambda_p)^2 N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \left(N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \right)^* \Delta(-2sB_0p_{\parallel}) q_{\Lambda_p\Lambda_p}^{-s-s} q_{\Lambda_q\Lambda_q}^{ss} \\ & + \\ & (k\Lambda + q\Lambda_q - p\Lambda_p)^2 N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \left(-N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \right)^* \Delta(2sB_0p_{\parallel}) q_{\Lambda_p\Lambda_p}^{-s-s} q_{\Lambda\Lambda}^{ss} \\ & + \\ & (k\Lambda + q\Lambda_q - p\Lambda_p)^2 N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \left(N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \right)^* \Delta(2sB_0p_{\parallel}) q_{\Lambda_p\Lambda_p}^{-s-s} q_{\Lambda_q\Lambda_q}^{ss} \} \\ & \delta_{pq,k} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

The symmetry between terms is recovered; further simplifications lead to:

$$\partial_t q_{\Lambda\Lambda}^{ss}(\mathbf{k}) = \frac{\epsilon}{16} \int \sum_{\Lambda_p, \Lambda_q} \left| N_{-k\ p\ q}^{\Lambda\Lambda_p\Lambda_q} \right|^2 (k\Lambda + q\Lambda_q - p\Lambda_p)^2 \quad (159)$$

$$(\Delta(-2sB_0p_{\parallel}) + \Delta(2sB_0p_{\parallel})) q_{\Lambda_p\Lambda_p}^{-s-s} (q_{\Lambda_q\Lambda_q}^{ss} - q_{\Lambda\Lambda}^{ss}) \delta_{pq,k} d\mathbf{p} d\mathbf{q}.$$

The long-time behavior is given by the Riemman-Lebesgue Lemma; one finds:

$$\begin{aligned} \partial_t q_{\Lambda\Lambda}^{ss}(\mathbf{k}) = & \quad (160) \\ \frac{\pi\epsilon}{16B_0} \int \sum_{\Lambda_p, \Lambda_q} \left(\frac{\sin \psi_k}{k} \right)^2 & (k\Lambda + p\Lambda_p + q\Lambda_q)^2 (k\Lambda + q\Lambda_q - p\Lambda_p)^2 \\ & q_{\Lambda_p\Lambda_p}^{-s-s} (q_{\Lambda_q\Lambda_q}^{ss} - q_{\Lambda\Lambda}^{ss}) \delta(p_{\parallel}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

Equations (160) are the wave kinetic equations for incompressible MHD turbulence at the level of three-wave interactions when helicities are absent and when equality between shear- and pseudo-Alfvén wave energies is assumed (see Section 4.5). We find the same equations as (72) since we have the relation $q_{\Lambda\Lambda}^{ss}(\mathbf{k}) = 4 q_{\Lambda}^s(\mathbf{k})$.

E Kinetic equations for the energies E and E_d

We introduce the expression (74) into (56) and we consider a state of zero helicities ($H_m(\mathbf{k}) = 0$ and $H_G(\mathbf{k}) = 0$). The kinetic equations for the energies E and E_d are

$$\begin{aligned} \partial_t \left\{ \begin{array}{l} E(\mathbf{k}) \\ E_d(\mathbf{k}) \end{array} \right\} = & \quad (161) \\ \frac{\pi \epsilon^2}{32 d_i^2 B_0^2} \int \sum_{\substack{\Lambda, \Lambda_p, \Lambda_q \\ s, s_p, s_q}} \left(\frac{\sin \psi_k}{k} \right)^2 \frac{(\Lambda k + \Lambda_p p + \Lambda_q q)^2 \left(1 - \xi_{\Lambda}^{-s^2} \xi_{\Lambda_p}^{-s_p^2} \xi_{\Lambda_q}^{-s_q^2} \right)^2}{(1 + \xi_{\Lambda}^{-s^2})(1 + \xi_{\Lambda_p}^{-s_p^2})(1 + \xi_{\Lambda_q}^{-s_q^2})} \\ & \left(\frac{\xi_{\Lambda_q}^{s_q} - \xi_{\Lambda_p}^{s_p}}{k_{\parallel}} \right)^2 \left\{ \frac{\xi_{\Lambda}^{-s} + 1}{\xi_{\Lambda}^{-s} - 1} \right\} \frac{\omega_{\Lambda}^s \omega_{\Lambda_p}^{s_p}}{\xi_{\Lambda}^{-s^2} + 1} \left[E(\mathbf{q}) + \left(\frac{\xi_{\Lambda_q}^{-s_q^2} + 1}{\xi_{\Lambda_q}^{-s_q^2} - 1} \right) E_d(\mathbf{q}) \right] \\ & \left[E(\mathbf{p}) - E(\mathbf{k}) + \left(\frac{\xi_{\Lambda_p}^{-s_p^2} + 1}{\xi_{\Lambda_p}^{-s_p^2} - 1} \right) E_d(\mathbf{p}) - \left(\frac{\xi_{\Lambda}^{-s^2} + 1}{\xi_{\Lambda}^{-s^2} - 1} \right) E_d(\mathbf{k}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}. \end{aligned}$$

F Compatibility with Galtier et al. (2000)

We start from equation (26) of Galtier et al. (2000). We assume that helicities are absent ($I^s(\mathbf{k}) = 0$) and that shear- and pseudo-Alfvén wave energies are identical. We introduce the following notations:

$$k_{\perp}^2 \Psi^s(\mathbf{k}) = e^s(\mathbf{k})/2, \quad (162)$$

and

$$k_{\perp}^2 k^2 \Phi^s(\mathbf{k}) = e^s(\mathbf{k})/2. \quad (163)$$

Then equation (26) writes:

$$\begin{aligned} \partial_t e^s(\mathbf{k}) = & \quad (164) \\ & \frac{\pi \epsilon^2}{4B_0} \int \left\{ \left(1 - \frac{(\mathbf{k} \times \kappa)^2}{L_{\perp}^2 k^2} + \frac{k_{\parallel}^2 \kappa_{\perp}^2}{L_{\perp}^2 k^2}\right) e^s(\mathbf{L}) - \left(1 - \frac{(\mathbf{k} \times \kappa)^2}{k_{\perp}^2 L^2} + \frac{k_{\parallel}^2 \kappa_{\perp}^2}{k_{\perp}^2 L^2}\right) e^s(\mathbf{k}) \right. \\ & + \\ & \left. \left(1 - \frac{k_{\parallel}^2 \kappa^2 \cos^2 \psi_k}{L_{\perp}^2 k^2}\right) e^s(\mathbf{L}) - \left(1 - \frac{k_{\parallel}^2 \kappa^2 \cos^2 \psi_L}{k_{\perp}^2 L^2}\right) e^s(\mathbf{k}) \right\} \\ & \{ (\mathbf{k} \times \kappa)^2 - k_{\parallel}^2 \kappa_{\perp}^2 \} \frac{e^{-s}(\kappa)}{\kappa_{\perp}^2} + k_{\parallel}^2 e^{-s}(\kappa) \} \delta(\kappa_{\parallel}) \delta_{k,\kappa L} d\kappa d\mathbf{L}. \end{aligned}$$

We remind that the angle ψ_k refers to the angle opposite to the 3D vector \mathbf{k} in the triangle defined by $\mathbf{k} = \mathbf{L} + \kappa$. (In Galtier et al. (2000), angles are introduced in reference to 2D wavevectors.) Some manipulations lead to:

$$\begin{aligned} \partial_t e^s(\mathbf{k}) = & \quad (165) \\ & \frac{\pi \epsilon^2}{4B_0} \int \left\{ \left(2 - \frac{\kappa_{\perp}^2 \sin^2 \psi_L}{L_{\perp}^2} + \frac{k_{\parallel}^2 \kappa_{\perp}^2}{L_{\perp}^2 k^2} - \frac{k_{\parallel}^2 \kappa_{\perp}^2 \cos^2 \psi_k}{L_{\perp}^2 k^2}\right) e^s(\mathbf{L}) \right. \\ & - \\ & \left. \left(2 - \frac{(k_{\perp}^2 \kappa_{\perp}^2 \sin^2 \psi_L}{k_{\perp}^2 L^2} + \frac{k_{\parallel}^2 \kappa_{\perp}^2}{k_{\perp}^2 L^2} - \frac{k_{\parallel}^2 \kappa_{\perp}^2 \cos^2 \psi_L}{k_{\perp}^2 L^2}\right) e^s(\mathbf{k}) \right\} \\ & k^2 \sin^2 \psi_L e^{-s}(\kappa) \delta(\kappa_{\parallel}) \delta_{k,\kappa L} d\kappa d\mathbf{L}, \end{aligned}$$

that can be written as:

$$\begin{aligned} \partial_t e^s(\mathbf{k}) = & \quad (166) \\ & \frac{\pi \epsilon^2}{4B_0} \int \{ (1 + \cos^2 \psi_{\kappa}) L^2 \sin^2 \psi_k e^{-s}(\kappa) (e^s(\mathbf{L}) - e^s(\mathbf{k})) \delta(\kappa_{\parallel}) \delta_{k,\kappa L} d\kappa d\mathbf{L}. \end{aligned}$$

Equations (166) are the wave kinetic equations for incompressible MHD turbulence when helicities are absent and when equality between shear- and pseudo-Alfvén wave energies is assumed. The addition over the index s will give the wave kinetic equations for the total energy.

G Nonlocal WCC interactions

We shall consider the strongly nonlocal interactions of two ion cyclotron waves at small scales on one whistler wave at large scales. In other words, it means that the whistler wave will be supported by the wavevector \mathbf{k} , with $k \ll p, q$. This type of interaction is the most interesting since smaller scales may be reached more easily by ion cyclotron waves. For such a limit, the master equations (75) reduce to

$$\partial_t \left\{ \begin{matrix} E^V(\mathbf{k}) \\ E^B(\mathbf{k}) \end{matrix} \right\} = \quad (167)$$

$$\frac{\pi \epsilon^2}{8 d_i^4} \int \sum_{s, s_p, s_q} \left(\frac{\sin \psi_k}{k} \right)^2 \left(\frac{s_q/q - s_p/p}{k_{\parallel}} \right)^2 (s_p p + s_q q)^2 \frac{p q^2 k_{\parallel} p_{\parallel}}{k^3} \left\{ \begin{matrix} \frac{1}{d_i^2 k^2} \\ 1 \end{matrix} \right\}$$

$$E^V(\mathbf{q}) \left[E^V(\mathbf{p}) - E^B(\mathbf{k}) \right] \delta(\Omega_{k,pq}) \delta_{k,pq} d\mathbf{p} d\mathbf{q}.$$

As expected, at leading order the whistler wave is described by the magnetic energy. Contrary to the other cases studied before, a non trivial dynamics happens as long as a discrepancy exists between the (kinetic) energy of the ion cyclotron waves and the (magnetic) energy of the whistler wave.

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